Michigan Math. J. 61 (2012), 227–237

Outer Automorphisms of Algebraic Groups and Determining Groups by Their Maximal Tori

Skip Garibaldi

One goal of this paper is to prove Theorem 20 below, which completes some of the main results in the remarkable paper [PRa1] by Gopal Prasad and Andrei Rapinchuk. For example, combining their Theorem 7.5 with our Theorem 20 gives the following statement.

Theorem 1. *Let G_1 and G_2 be connected, absolutely simple algebraic groups over a number field K that have the same K-isomorphism classes of maximal K-tori. Then*:

(1) *G_1 and G_2 have the same Killing–Cartan type* (*and even the same quasi-split inner form*) *or one has type B_n and the other has type C_n*;
(2) *if G_1 and G_2 are isomorphic over an algebraic closure of K and are not of type A_n for $n \geq 2$, D_{2n+1}, or E_6, then G_1 and G_2 are K-isomorphic.*

This result is mostly proved in [PRa1], except that paper omits types D_{2n} for $2n \geq 4$ in (2). Our Theorem 20 gives a new proof of the $2n \geq 6$ case (treated by Prasad and Rapinchuk in a later paper [PRa2, Sec. 9]) and settles the last remaining case of groups of type D_4. Note that in Theorem 1(2), types A_n, D_{2n+1}, and E_6 are genuine exceptions by [PRa1, 7.6].

Similarly, combining our Theorem 20 with the arguments in [PRa1] implies that their Theorems 4, 8.16, and 10.4 remain true if one deletes "D_4" from their statements—that is, the conclusions of those theorems regarding weak commensurability, locally symmetric spaces, and so on hold also for groups of type D_4.

We mention the following specific result as an additional illustration. For a Riemannian manifold M, write $\mathbb{Q}L(M)$ for the set of rational multiples of lengths of closed geodesics of M.

Theorem 2. *Let M_1 and M_2 be arithmetic quotients of real hyperbolic space $\mathbf{H}^n$ for some $n \not\equiv 1 \bmod 4$. If $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$, then M_1 and M_2 are commensurable* (*i.e., M_1 and M_2 have a common finite-sheeted cover*).

The converse holds with no restriction on n; see [PRa1, Cor. 8.7]. The theorem itself holds for $n = 2$ by [Re]; for $n = 3$ by [CHLRe]; and for $n = 4, 6, 8, \dots$ and $n = 11, 15, 19, \dots$ by [PRa1, Cor. 8.17] (which relies on [PRa2]). The last remaining case, $n = 7$, follows from Theorem 20 (to follow) and arguments as in [PRa1].

Received August 10, 2010. Revision received September 15, 2011.

The conclusion of Theorem 2 is false for $n = 5, 9, 13, \ldots$ by Construction 9.15 in [PRa1].

The other goal of this paper is to prove Theorem 11, which addresses the more general setting of a semisimple algebraic group G over an arbitrary field k. That theorem gives a cohomological criterion for the existence of outer automorphisms of G—in other words, for the existence of k-points on nonidentity components of $\mathrm{Aut}(G)$. This criterion and the examples we give of when it holds make up the bulk of the proof of Theorem 20, which concerns groups over global fields.

NOTATION. A *global field* is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$ for some prime p. A (non-Archimedean) *local field* is the completion of a global field with respect to a discrete valuation (i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime p).

We write $H^d(k, G)$ for the dth flat (fppf) cohomology set $H^d(\mathrm{Spec}\, k, G)$ when G is an algebraic affine group scheme over a field k. In case G is smooth, it is the same as the Galois cohomology set $H^d(\mathrm{Gal}(k), G(k_{\mathrm{sep}}))$, where k_{sep} denotes a separable closure of k and $\mathrm{Gal}(k)$ denotes the group of k-automorphisms of k_{sep}.

We refer to [KMRT; PlRa; Sp2] for general background on semisimple algebraic groups. Such a group G is an *inner form* of G' if there is a class $\gamma \in H^1(k, \bar{G})$, for $\bar{G}$ the adjoint group of G, such that G' is isomorphic to G twisted by γ. We write G_γ for the group G twisted by the cocycle γ, following the typesetter-friendly notation of [KMRT, p. 387] instead of Serre's more logical ${}_\gamma G$. We say simply that G is *inner* or *of inner type* if it is an inner form of a split group; if G is not inner then it is *outer.*

For a group scheme D of multiplicative type, we put D^* for its Cartier dual $\mathrm{Hom}(D, \mathbb{G}_m)$.

1. Background: The Tits Algebras Determine the Tits Class

Fix a semisimple algebraic group G over a field k. Its simply connected cover $\tilde{G}$ and adjoint group $\bar{G}$ fit into an exact sequence

$$1 \longrightarrow Z \longrightarrow \tilde{G} \xrightarrow{q} \bar{G} \longrightarrow 1, \tag{3}$$

where Z denotes the (scheme-theoretic) center of $\tilde{G}$. Write $\delta\colon H^1(k, \bar{G}) \to H^2(k, Z)$ for the corresponding coboundary map.

There is a unique element $\nu_G \in H^1(k, \bar{G})$ such that the twisted group $\bar{G}_{\nu_G}$ is quasi-split [KMRT, 31.6], and the *Tits class* t_G of G is defined to be $t_G := -\delta(\nu_G) \in H^2(k, Z)$. The element t_G depends only on the isogeny class of G.

For $\gamma \in H^1(k, \bar{G})$, the center of the twisted group $\tilde{G}_\gamma$ is naturally identified with (and not merely isomorphic to) Z, and a standard twisting argument shows that

$$t_{G_\gamma} = t_G + \delta(\gamma). \tag{4}$$

EXAMPLE 5. If G itself is quasi-split, then $t_G = 0$ and for every $\gamma \in H^1(k, \bar{G})$ we have $t_{G_\gamma} = \delta(\gamma)$.

DEFINITION 6 (Tits algebra). A *Tits algebra* of G is an element

$$\chi(t_G) \in H^2(k(\chi), \mathbb{G}_m) \quad \text{for} \;\; \chi \in Z^*,$$

where $k(\chi)$ denotes the subfield of k_{sep} of elements fixed by the stabilizer of χ in $\text{Gal}(k)$; that is, $k(\chi)$ is the smallest separable extension of k such that χ is fixed by $\text{Gal}(k(\chi))$.

The Tits algebras of G can be interpreted as follows. Each maximal k-torus $\tilde{T}$ of $\tilde{G}$ contains the center Z, so every character $\lambda \in \tilde{T}^*$ gives a well-defined Tits algebra $\lambda(t_G) := \lambda|_Z(t_G)$. This $\lambda(t_G)$ measures the failure of the irreducible representation of $\tilde{G}$ with highest weight λ—which is defined over k_{sep}—to be defined over k [Ti, Sec. 7]. Roughly speaking, a typical example of a Tits algebra is provided by the even Clifford algebra of the special orthogonal group of a quadratic form (see e.g. [KMRT, Sec. 27]).

Obviously, the Tits class t_G determines the Tits algebras $\chi(t_G)$ for all χ. The converse also holds, as is shown next.

PROPOSITION 7. *The natural map*

$$\prod \chi : \; H^2(k, Z) \to \prod_{\chi \in Z^*} H^2(k(\chi), Z) \tag{8}$$

is injective.

This proposition can probably be viewed as folklore. I learned it from Alexander Merkurjev and Anne Quéguiner.

Proof of Proposition 7. The claim depends only on Z, so we may replace $\tilde{G}$ with $\tilde{G}_{\nu_G}$ and so assume that $\tilde{G}$ is quasi-split. We pick a maximal k-torus $\tilde{T}$ contained in a Borel k-subgroup B of $\tilde{G}$. This determines a set Δ of simple roots of $\tilde{G}$ with respect to $\tilde{T}$ such that the natural Galois action on $\tilde{T}^*$ permutes Δ; in other words, it coincides with the $*$-action.

Recall that $\sigma \in \text{Gal}(k)$ acts naturally on $\lambda \in \tilde{T}^*$ via $({}^\sigma\lambda)(t) = \sigma(\lambda(\sigma^{-1}(t)))$ for $t \in \tilde{T}(k_{\text{sep}})$. Typically, this action does not leave Δ, equivalently B, invariant and one chooses $g_\sigma \in N_{\tilde{G}}(\tilde{T})(k_{\text{sep}})$ such that $\sigma(B) = \text{Int}(g_\sigma)(B)$. The $*$*-action* of $\text{Gal}(k)$ on $\tilde{T}$ is defined by $\sigma * \lambda := {}^\sigma\lambda \circ \text{Int}(g_\sigma)$; it normalizes Δ. In our case, $\sigma(B) = B$ and so we may take g_σ to be the identity.

We fix a set S of representatives of the $\text{Gal}(k)$-orbits in Δ and write α_s (resp., λ_s) for the simple root (resp., fundamental dominant weight) corresponding to $s \in S$. Because $\tilde{G}$ is simply connected,

$$\tilde{T} \cong \prod_{s \in S} R_{k(\lambda_s)/k}(\text{im}\, h_{\alpha_s}),$$

where h_{α_s} denotes the homomorphism $\mathbb{G}_m \to \tilde{T}$ corresponding to the coroot $\alpha_s^\vee$ [St, Cor., p. 44] and $k(\lambda_s)$ is the field of definition of λ_s (equivalently, α_s by our choice of B and $\tilde{T}$).

The product of the compositions

$$Z \xrightarrow{\lambda_s|_Z} R_{k(\lambda_s|_Z)/k}(\mathbb{G}_m) \hookrightarrow R_{k(\lambda_s)/k}(\mathbb{G}_m) \xrightarrow{h_{\alpha_s}} R_{k(\lambda_s)/k}(\operatorname{im} h_{\alpha_s})$$

for $s \in S$ is the inclusion of Z in $\tilde{T}$. It follows that the kernel of $\prod \chi$ is contained in the kernel of $H^2(k, Z) \to H^2(k, \tilde{T})$.

In the short exact sequence $1 \to Z \to \tilde{T} \to \tilde{T}/Z \to 1$, the set Δ is a basis for the lattice $(\tilde{T}/Z)^*$; hence $H^1(k, \tilde{T}/Z)$ is zero and the map $H^2(k, Z) \to H^2(k, \tilde{T})$ is injective. □

2. Outer Automorphisms of Semisimple Groups

We continue to consider a semisimple linear algebraic group G over a field k. In case G is split, there is a natural map $\alpha\colon \operatorname{Aut}(G) \to \operatorname{Aut}(\Delta)$ for Δ the Dynkin diagram of G; see [Sp2, Sec. 16.3]. In this section, we will define α also in the case where G is nonsplit and ask:

$$\textit{Is } \alpha\colon \operatorname{Aut}(G)(k) \to \operatorname{Aut}(\Delta)(k) \textit{ surjective?} \tag{9}$$

One obstruction to α being surjective can come from the fundamental group—this is clear from considering the case where G is the split group SO_8 of type D_4—so we assume that G is simply connected. (One could equivalently assume that G is adjoint.) Then, still assuming that G is split, α fits into an exact sequence

$$1 \longrightarrow \bar{G} \xrightarrow{\varepsilon} \operatorname{Aut}(G) \xrightarrow{\alpha} \operatorname{Aut}(\Delta) \longrightarrow 1, \tag{10}$$

where $\bar{G}$ denotes the adjoint group of G (see [Sp2]).

We claim that the maps in sequence (10) are defined and that the sequence is exact for arbitrary semisimple simply connected G. Indeed, such a G is obtained by twisting a split simply connected group G' by a cocycle $z \in Z^1(k, \operatorname{Aut}(G'))$. Starting with a version of (10) involving the split group G', we may twist by z and obtain sequence (10) for nonsplit G. Note that the resulting group scheme $\operatorname{Aut}(\Delta)$ is finite étale but not necessarily constant: $\operatorname{Gal}(k)$ acts on $\operatorname{Aut}(\Delta)(k_{\mathrm{sep}})$ via the $*$-action. (One can see this by reducing to the case where G is quasi-split, where it can be checked directly.) Furthermore, the sequence identifies $\operatorname{Aut}(\Delta)$ with the group of connected components of $\operatorname{Aut}(G)$, so Question 9 is the same as: *Does every connected component of* $\operatorname{Aut}(G) \times k_{\mathrm{sep}}$ *that is defined over k necessarily have a k-point?*

The Tits class provides an obstruction to the surjectivity of α, as we now explain. There is a commutative diagram

$$\begin{array}{ccc} \operatorname{Aut}(G) & \xrightarrow{\alpha} & \operatorname{Aut}(\Delta) \\ \downarrow & \swarrow & \\ \operatorname{Aut}(Z), & & \end{array}$$

where Z denotes the center of G and the diagonal arrow comes from the natural action of $\mathrm{Aut}(\Delta)$ on the coroot lattice. Hence $\mathrm{Aut}(\Delta)(k)$ acts on $H^2(k, Z)$ and we have the following result.

THEOREM 11. *Recall that G is assumed semisimple and simply connected. Then there is an inclusion*

$$\mathrm{im}[\alpha\colon \mathrm{Aut}(G)(k) \to \mathrm{Aut}(\Delta)(k)] \subseteq \{\pi \in \mathrm{Aut}(\Delta)(k) \mid \pi(t_G) = t_G\}. \tag{12}$$

Furthermore, the following are equivalent:

(a) *equality holds in* (12);
(b) *the sequence* $H^1(k, Z) \to H^1(k, G) \to H^1(k, \mathrm{Aut}(G))$ *is exact*;
(c) $\ker \delta \cap \ker[H^1(k, \bar{G}) \to H^1(k, \mathrm{Aut}(\bar{G}))] = 0$.

Proof. We consider the interlocking exact sequences

$$\begin{array}{ccccccc} & & & & H^1(k,Z) & & \\ & & & & \downarrow & & \\ & & & & H^1(k,G) & & \\ & & & & \downarrow q & & \\ \mathrm{Aut}(G)(k) & \xrightarrow{\alpha} & \mathrm{Aut}(\Delta)(k) & \xrightarrow{\beta} & H^1(k,\bar{G}) & \xrightarrow{\varepsilon} & H^1(k,\mathrm{Aut}(G)) \\ & & & & \downarrow \delta & & \\ & & & & H^2(k,Z), & & \end{array}$$

where the horizontal sequence comes from (10) and the vertical sequence comes from (3). The crux is to prove that

$$\pi(t_{G_{\beta(\pi)}}) = t_G \quad \text{for } \pi \in \mathrm{Aut}(\Delta)(k). \tag{13}$$

Since $\bar{G}$ and $\mathrm{Aut}(G)$ are smooth, we may view their corresponding H^1 as Galois cohomology. Put $\gamma := \beta(\pi)$, so $\gamma_\sigma = f^{-1}\,{}^{\sigma}\!f$ for some $f \in \mathrm{Aut}(G)(k_{\mathrm{sep}})$ and every $\sigma \in \mathrm{Gal}(k)$. The group G_γ has the same k_{sep}-points as G but has a different Galois action $\circ$, given by $\sigma \circ g = \gamma_\sigma \sigma g$ for $g \in G(k_{\mathrm{sep}})$ and $\sigma \in \mathrm{Gal}(k)$, where juxtaposition denotes the usual Galois action on G.

The map f gives a k-isomorphism $G_\gamma \xrightarrow{\sim} G$. Sequence (3) gives a commutative diagram

$$\begin{array}{ccc} H^1(k,\bar{G}_\gamma) & \xrightarrow{\delta_\gamma} & H^2(k,Z) \\ f\downarrow & & f\downarrow \\ H^1(k,\bar{G}) & \xrightarrow{\delta} & H^2(k,Z). \end{array}$$

Let $\eta \in Z^1(k, \bar{G}_\gamma)$ be a 1-cocycle representing ν_{G_γ}. Then $f(\eta)$ is a 1-cocycle in $Z^1(F, \bar{G})$ and f is a k-isomorphism $f\colon (G_\gamma)_\eta \xrightarrow{\sim} G_{f(\eta)}$. Since $(G_\gamma)_\eta$ is k-quasi-split, we have $f(\nu_{G_\gamma}) = f(\eta) = \nu_G$. The commutativity of the diagram gives $f(t_{G_\gamma}) = t_G$, proving (13).

It follows that $\pi \in \mathrm{Aut}(\Delta)(k)$ satisfies $\pi(t_G) = t_G$ if and only if $t_{G_{\beta(\pi)}} = t_G$, if and only if $\delta(\beta(\pi)) = 0$. That is, in (12), the left side is $\ker \beta$ and the right side is $\ker \delta\beta$, which makes the inclusion in (12) and the equivalence of (a) and (c) obvious. Statement (b) says that $\ker \varepsilon q = \ker q$ (i.e., $\ker \varepsilon \cap \operatorname{im} q = 0$), which is (c). $\square$

It is easy to find nonsimple groups (even over $\mathbb{R}$) for which the inclusion (12) is proper, because the Tits index also provides an obstruction to equality. Indeed, if G is the product of the split and the compact real forms of G_2, then the image of α is the identity but the right side of (12) is $\mathbb{Z}/2\mathbb{Z}$. We now slightly modify this example to show that the Tits algebras and Tits index are not the only obstructions to equality in (12), even over a number field.

EXAMPLE 14. Fix a prime p and write x_1, x_2 for the two square roots of p in $k := \mathbb{Q}(\sqrt{p})$. For $i = 1, 2$, let H_i be the group of type G_2 associated with the 3-Pfister quadratic form $\phi_i := \langle\langle -1, -1, x_i\rangle\rangle$. For $G = H_1 \times H_2$, the Tits index is

and the right side of (12) is $\mathbb{Z}/2\mathbb{Z}$, but H_1 is not isomorphic to H_2 and so no k-automorphism of G interchanges the two components.

Nonetheless, we now give several examples of equality holding in (12).

EXAMPLE 15. If G is quasi-split then α maps $\mathrm{Aut}(G)(k)$ onto $\mathrm{Aut}(\Delta)(k)$ by [SGA3, XXIV.3.10] or [KMRT, 31.4], so equality holds in (12).

EXAMPLE 16. If $H^1(k, G) = 0$, then trivially Theorem 11(b) holds. That is, (a)–(c) hold for every semisimple simply connected G if k is local (by Kneser–Bruhat–Tits), global with no real embeddings (Kneser–Harder–Chernousov), or the function field of a complex surface (de Jong–He–Starr–Gille), and conjecturally if the cohomological dimension of k is at most 2 (Serre).

EXAMPLE 17. Suppose G is absolutely almost simple (and simply connected). Conditions (a)–(c) of the proposition hold trivially if $\mathrm{Aut}(\Delta)(k) = 1$, in particular if G is not of type A, D, or E_6 or if G has type ${}^6D_4$. Conditions (a)–(c) also hold in the following four cases.

Case (i): G is of inner type. If G is of inner type A ($n \geq 2$), then $\mathrm{Aut}(\Delta) = \mathbb{Z}/2\mathbb{Z}$ and the nontrivial element π acts via $z \mapsto z^{-1}$ on Z; hence $\pi(t_G) = -t_G$. If $2t_G = 0$, then G is $\mathrm{SL}_1(D)$ for D a central simple algebra of degree $n+1$ such that there is an anti-automorphism σ of D; therefore, $g \mapsto \sigma(g)^{-1}$ is a k-automorphism of G mapping to π. (By a theorem of Albert [Sch, Thm. 8.8.4] one can even arrange for σ to have order 2 and thus for this automorphism of G to have order 2.)

Next let G be of type ${}^1D_n$ for $n \geq 5$ and suppose that the nonidentity element $\pi \in \mathrm{Aut}(\Delta)(k)$ fixes the Tits class t_G. The group G is isomorphic to $\mathrm{Spin}(A, \sigma, f)$ for some central simple k-algebra A of degree $2n$ and quadratic pair (σ, f) on A such that the even Clifford algebra $C_0(A, \sigma, f)$ is isomorphic to a direct product $C_+ \times C_-$ of central simple algebras. Since π fixes the Tits class, the algebras C_+ and C_- are isomorphic. The equation $[A] + [C_+] - [C_-] = 0$ holds in the Brauer group of k by [KMRT, 9.12] (alternatively, because the cocenter is an abelian group of order 4). Therefore, A is split. Let $\phi \in \mathrm{O}(A, \sigma, f)(k)$ be a hyperplane reflection as in [KMRT, 12.13]; it does not lie in the identity component of $\mathrm{O}(A, \sigma, f)$. The automorphism of $\mathrm{SO}(A, \sigma, f)$ given by $g \mapsto \phi g \phi^{-1}$ lifts to an automorphism of $\mathrm{Spin}(A, \sigma, f)$ that is outer (i.e., that induces the automorphism π on Δ).

To recap: Given a nonzero $\pi \in \mathrm{Aut}(\Delta)(k)$ that preserves the Tits class, we deduced that A is split and thus that (A, σ, f) has an improper isometry. Conversely, [KMRT, 13.38(2)] shows: if (A, σ) has an improper isometry, then A is split and obviously such an automorphism π exists.

For the remaining cases, we simply point out that an outer automorphism of order 3 in the D_4 case exists when $t_G = 0$ by triality [SpV, 3.6.3, 3.6.4], and an outer automorphism of order 2 exists in the E_6 case when $t_G = 0$ is provided by the "standard automorphism" of a J-structure [Sp1, p. 150].

Case (ii): G is the special unitary group of a hermitian form relative to a separable quadratic extension K/k (i.e., G is of type ${}^2A_n$ and $\mathrm{res}_{K/k}(t_G)$ is zero in $H^2(K, Z)$). We leave the details in this case as an exercise.

Case (iii): k is real closed. By the previous cases, we may assume that G has type ${}^2D_n$ (for $n \geq 4$) or ${}^2E_6$. For type ${}^2D_n$, $\mathrm{Aut}(\Delta)(k) = \mathbb{Z}/2\mathbb{Z}$ (also for $n = 4$) and G is the spin group of a quadratic form by [KMRT, 9.14], so a hyperplane reflection gives the desired k-automorphism.

If G has type ${}^2E_6$, then combining the arguments on pages 37, 38, 119, and 120 in [J] shows that the (outer) automorphism of the Lie algebra Jacobson denotes by t is defined over k.

Case (iv): k is global. Let $\gamma \in H^1(k, \bar{G})$ lie in $\ker \delta \cap \ker \varepsilon$ for δ, ε as in the proof of Theorem 11. At every completion k_v of k, $\mathrm{res}_{k_v/k}(\gamma)$ lies in the kernel of the maps from $H^1(k_v, \bar{G})$ to $H^2(k_v, Z)$ and $H^1(k_v, \mathrm{Aut}(\bar{G}))$, so $\mathrm{res}_{k_v/k}(\gamma)$ is zero by (iii) (for v real) and Example 16 (for v finite). The Hasse principle for adjoint groups [PlRa, Thm. 6.22] gives that γ is zero; hence (c) holds.

I don't know any examples of absolutely almost simple G where conditions (a)–(c) fail. Furthermore, in all of the examples given here, every π from the right side of (12) is not only of the form $\alpha(f)$ for some $f \in \mathrm{Aut}(G)(k)$, but one can even pick f to have the same order as π.

We illustrate the foregoing results in the case of an arbitrary group of type ${}^2A_n$.

Example 18. Let G_0 be an absolutely almost simple algebraic group of type ${}^2A_n$ over a field k. Its simply connected cover G is isomorphic to $\mathrm{SU}(B, \tau)$ for some central simple algebra B with center a separable quadratic extension L/k and τ a unitary L/k-involution on L [KMRT, 26.9].

If G_0 has an outer automorphism defined over k, then B has exponent 1 or 2. Indeed, every k-automorphism of G_0 gives an L-automorphism of $\mathrm{SU}(B,\tau)\times L$, which is $\mathrm{SL}_1(B)$. The claim follows by Example 17(i).

Conversely, *if k is a global field and B has exponent (equivalently, index) 1 or 2, then G has an outer automorphism defined over k.* Indeed, the hypothesis on the exponent of B gives that the nonidentity element $\pi \in \mathrm{Aut}(\Delta)(k)$ fixes t_G; hence, by Example 17(iv), π is the image of some element of $\mathrm{Aut}(G)(k)$.

3. Groups of Type D_{even} over Local Fields

The main point of this section is to prove the following lemma.

LEMMA 19. *Let G be an adjoint semisimple group over a field k, and fix a maximal k-torus T in G. If z_1, z_2 are in the image of the map $H^1(k,T) \to H^1(k,G)$ such that*

(1) *G_{z_1} and G_{z_2} are both quasi-split or*
(2) *T contains a maximal k-split torus in both G_{z_1} and G_{z_2} and*
   (a) *k is real closed or*
   (b) *k is a (non-Archimedean) local field and G has type D_{2n} for some $n \geq 2$,*

then $z_1 = z_2$.

Proof. For short, we write G_i for G_{z_i}. In case (1), the uniqueness of the class $\nu_G \in H^1(k,G)$ such that G_{ν_G} is quasi-split (already used in Section 1) gives that $z_1 = \nu_G = z_2$. So suppose (2) holds. As T is contained in both these groups, their Tits indexes are naturally identified over k. In particular, if one is quasi-split then so is the other, and we are done as in (1). So we assume that neither group is quasi-split.

In case (2)(a), where k is real closed, one immediately reduces to the case where G is absolutely simple. That case is trivial because the isomorphism class of an adjoint simple group is determined by its Tits index, so G_1 is isomorphic to G_2. The Tits index also determines the Tits algebras—see [Ti, pp. 211–212] for a recipe—and so, by Proposition 7, $\delta(z_1) = \delta(z_2)$. The claim now follows from Example 17(iii) and Theorem 11(c).

So assume for the remainder of the proof that (2)(b) holds. In particular, δ is injective. Number the simple roots of G_1 with respect to T as in [B]. If G_1 has type ${}^2D_4$, we take α_1 to be the root at the end of the Galois-fixed arm of the Tits index. Otherwise, we assign the numbering arbitrarily in case there is ambiguity (e.g., α_{2n-1} and α_{2n}). Note that G_1 cannot have type ${}^3D_4$ or ${}^6D_4$ because it is not quasi-split.

As $2\omega_i$ is in the root lattice for every i, the Tits algebras $\omega_i(t_{G_1})$ for $i = 2n-1, 2n$ define up to k-isomorphism a quaternion (Azumaya) algebra D over a quadratic étale k-algebra ℓ. By the exceptional isomorphism $D_2 = A_1 \times A_1$ and a Tits algebra computation, $\mathrm{PGL}_1(D)$ is isomorphic to $\mathrm{PSO}(M_2(H), \sigma, f)$ for H the quaternion algebra underlying $\omega_1(t_{G_1})$ and some quadratic pair (σ, f) such that the even Clifford algebra $C_0(\sigma, f)$ is isomorphic to D (cf. [KMRT, 15.9]). Appending $2n-2$

hyperbolic planes to (σ, f), we obtain a quadratic pair (σ_0, f_0) such that $C_0(\sigma_0, f_0)$ is Brauer-equivalent to D. We have thus constructed $\mathrm{PSO}(M_n(H), \sigma_0, f_0)$ so that it is isomorphic to G_z for some class $z \in H^1(k, G)$ with $\omega_i(\delta(z)) = \omega_i(\delta(z_1))$ for $i = 1, 2n-1, 2n$. Examining the root system of type D_{2n}, we find that restricting the three ω_i to the center of the simply connected cover of G gives all three nonzero elements of the cocenter [B, Sec. VI.2, Exer. 5a]; in other words, G_z and G_1 have the same Tits algebras. Hence Proposition 7 and the injectivity of δ imply that $z = z_1$, so G_1 is isotropic and its semisimple anisotropic kernel is a product of groups with Killing–Cartan type A_1. (We have just given a characteristic-free proof of Tsukamoto's theorem [Sch, 10.3.6], relying on the Bruhat–Tits result that δ is injective.)

The same argument applied to G_2 shows that it is also isotropic with the same kind of semisimple anisotropic kernel. Since the two groups have the same Tits index and there is a unique quaternion division algebra over each finite extension of k, it follows that G_1 and G_2 have the same Tits class—that is, $\delta(z_1) = \delta(z_2)$. □

In the statement of (2)(b), we cannot replace "D_{2n} for some $n \geq 2$" with "D_ℓ for some ℓ" because the claim fails for groups of type D_{odd}. This can been seen already for type $D_3 = A_3$: one can find $z_1, z_2 \in H^1(k, \mathrm{PGL}_4)$ such that G_1 and G_2 are both isomorphic to $\mathrm{Aut}(B)^\circ$ for a division algebra B of degree 4, but $\delta(z_1) = -\delta(z_2)$ in $H^2(k, \mu_4) = \mathbb{Z}/4\mathbb{Z}$. Adding hyperbolic planes as in the proof of the lemma gives a counterexample for all odd ℓ. This counterexample is visible in the proof: for groups G_1, G_2 of type D_ℓ with $\ell \geq 3$ and odd, the semisimple anisotropic kernels have Killing–Cartan type a product of the A_1 and an A_3; hence the very last sentence of the proof fails.

4. Groups of Type D_{even} over Global Fields

The following technical theorem concerning groups over a global field connects our Theorem 11 (about groups over an arbitrary field) with the results in [PRa1]. It implies [PRa2, Thm. 9.1].

Theorem 20. *Let G_1 and G_2 be adjoint groups of type D_{2n} for some $n \geq 2$ over a global field K such that G_1 and G_2 have the same quasi-split inner form*; *in other words, the smallest Galois extension of K over which G_1 is of inner type is the same as for G_2. If there exists a maximal torus T_i in G_i for $i = 1, 2$ such that*

(1) *there is a K_{sep}-isomorphism $\phi\colon G_1 \to G_2$ whose restriction to T_1 is a K-isomorphism $T_1 \to T_2$ and*
(2) *there is a finite set V of places of K such that*
  (a) *for all $v \notin V$, G_1 and G_2 are quasi-split over K_v and*
  (b) *for all $v \in V$, $(T_i)_{K_v}$ contains a maximal K_v-split torus of $(G_i)_{K_v}$,*

then G_1 and G_2 are isomorphic over K.

The hypotheses are what one obtains by assuming the existence of weakly commensurable arithmetic subgroups—see, for example, Theorems 1 and 6 and Remark 4.4

(and especially p. 156) in [PRa1]. Note that the groups appearing in the theorem can be trialitarian (i.e., of type ${}^3D_4$ or ${}^6D_4$). We remark that Allison gave an isomorphism criterion with very different hypotheses in [A, Thm. 7.7].

Proof of Theorem 20. Write G for the unique adjoint quasi-split group that is an inner form of G_1 and G_2. According to Steinberg [PlRa, pp. 338–339], there is a K_{sep}-isomorphism $\psi_2 \colon G_2 \to G$ whose restriction to T_2 is defined over K. We put $\psi_1 := \psi_2\phi$ and $T := \psi_2(T_2) = \psi_1(T_1)$. Then G_i is isomorphic to G twisted by the 1-cocycle $\sigma \mapsto \psi_i({}^\sigma\psi_i)^{-1}$. But this 1-cocycle consists of elements of $\mathrm{Aut}(G)$ that fix T elementwise and thus belong to T itself. That is, for $i = 1, 2$, there is a cocycle z_i in the image of $H^1(K, T) \to H^1(K, G)$ such that G_i is isomorphic to G twisted by z_i. (This argument uses neither that K is a number field nor that G_1 and G_2 have type D_{2n}, so roughly speaking it applies generally to the situation where G_1 and G_2 share a maximal torus over the base field—more precisely, to the situation arising in [PRa1, Rem. 4.4].)

Now Lemma 19 gives that $\mathrm{res}_{K_v/K}(z_1) = \mathrm{res}_{K_v/K}(z_2)$ for every v, so $z_1 = z_2$ by the Kneser–Harder–Hasse principle [PlRa, Thm. 6.22] and G_1 is isomorphic to G_2 over K. □

ACKNOWLEDGMENTS. My research was partially supported by NSF grant DMS-0653502 and NSA grant H98230-11-1-0178. I thank the University of Virginia for its hospitality during a visit there where this work began, and Andrei Rapinchuk, Gopal Prasad, and Emily Hamilton for their comments and suggestions.

References

[A] B. N. Allison, *Lie algebras of type D_4 over number fields,* Pacific J. Math. 156 (1992), 209–250.

[B] N. Bourbaki, *Lie groups and Lie algebras,* Elem. Math. (Berlin), chaps 4–6, Springer-Verlag, Berlin, 2002.

[CHLRe] T. Chinburg, E. Hamilton, D. D. Long, and A. W. Reid, *Geodesics and commensurability classes of arithmetic hyperbolic 3-manifolds,* Duke Math. J. 145 (2008), 25–44.

[SGA3] M. Demazure and A. Grothendieck, *Schémas en groupes III: Structure des schemas en groupes reductifs,* Lecture Notes in Math., 153, Springer-Verlag, Berlin, 1970.

[J] N. Jacobson, *Exceptional Lie algebras,* Lecture Notes in Pure and Appl. Math., 1, Dekker, New York, 1971.

[KMRT] M.-A. Knus, A. S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions,* Amer. Math. Soc. Colloq. Publ., 44, Amer. Math. Soc., Providence, RI, 1998.

[PlRa] V. P. Platonov and A. Rapinchuk, *Algebraic groups and number theory,* Pure Appl. Math., 139, Academic Press, Boston, MA, 1994.

[PRa1] G. Prasad and A. S. Rapinchuk, *Weakly commensurable arithmetic groups and isospectral locally symmetric spaces,* Inst. Hautes Études Sci. Publ. Math. 109 (2009), 113–184.

[PRa2] ———, *Local-global principles for embedding of fields with involution into simple algebras with involution,* Comment. Math. Helv. 85 (2010), 583–645.

[Re] A. W. Reid, *Isospectrality and commensurability of arithmetic hyperbolic 2- and 3- manifolds,* Duke Math. J. 65 (1992), 215–228.

[Sch] W. Scharlau, *Quadratic and hermitian forms,* Grundlehern Math. Wiss., 270, Springer-Verlag, Berlin, 1985.

[Sp1] T. A. Springer, *Jordan algebras and algebraic groups,* Ergeb. Math. Grenzgeb. (3), 75, Springer-Verlag, New York, 1973.

[Sp2] ———, *Linear algebraic groups,* 2nd ed., Progr. Math., 9, Birkhäuser, Boston, 1998.

[SpV] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan algebras and exceptional groups,* Springer-Verlag, Berlin, 2000.

[St] R. Steinberg, *Lectures on Chevalley groups,* Yale Univ. Press, New Haven, CT, 1968.

[Ti] J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque,* J. Reine Angew. Math. 247 (1971), 196–220.

Department of Mathematics and Computer Science
Emory University
Atlanta, GA 30322

skip@member.ams.org