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The γ-filtration and the Rost invariant

By *Skip Garibaldi* at Atlanta and *Kirill Zainoulline* at Ottawa

Abstract. Grothendieck studied two filtrations – the γ and topological filtrations – on the ring $K_0(X)$ for a projective variety X, together with the associated graded rings. He gave a necessary and sufficient condition for the graded ring $\oplus_i \gamma^i/\gamma^{i+1}$ to have non-zero torsion elements when X is the variety of Borel subgroups of a simply connected semisimple algebraic group G over an algebraically closed field. We sharpen his observation by calculating explicitly the torsion in γ^2/γ^3, and we do this under weaker hypotheses on G. We apply this result to describing the torsion in the group $\mathrm{CH}^3(X)$ of codimension-3 cycles on X, and providing an extension of the Rost invariant.

Introduction

The core result in this paper extends and sharpens some observations of Grothendieck from SGA6 [5, §XIV.4.5] concerning the ring $K_0(X)$ of coherent sheaves on a projective variety X. He studied the γ-filtration $\gamma^0 \supseteq \gamma^1 \supseteq \cdots$ and the topological filtration $\tau^0 \supseteq \tau^1 \supseteq \cdots$ on $K_0(X)$ and compared the corresponding graded rings $\oplus_{i\geq 0}\gamma^{i/i+1}$ and $\oplus_{i\geq 0}\tau^{i/i+1}$ with the Chow ring $\mathrm{CH}(X)$. There are natural homomorphisms between these rings, which by Riemann–Roch become isomorphisms after tensoring with $\mathbb{Q}$, see [11, §15.3]. Therefore, to compare them one should examine their torsion subgroups. In case X is the variety of Borel subgroups of a semisimple linear algebraic group G over an algebraically closed field k, Grothendieck observed in [5, p. 678] that $\oplus\tau^{i/i+1}$ and $\mathrm{CH}(X)$ are torsion-free and so isomorphic, but $\oplus\gamma^{i/i+1}$ *is torsion-free if and only if the simply connected cover of* G *is isomorphic to a product of copies of* SL_n *and* Sp_{2n} *for various* n. In particular, $\oplus\gamma^{i/i+1}$ has non-zero torsion for $G = \mathrm{SO}_n$ with $n \geq 4$ or for any exceptional group.

We make Grothendieck's result more precise by explicitly calculating the torsion in some cases. (We also ease the hypotheses on G and k. We allow k to be arbitrary, but assume that G is split, i.e., that G contains a maximal k-torus that is split over k. This is automatic in case k is algebraically closed, so is indeed a weakening of Grothendieck's hypotheses.) Specifically, one knows that

$$\gamma^{0/1} \cong \mathrm{CH}^0(X) \cong \mathbb{Z} \quad \text{and} \quad \gamma^{1/2} \cong \mathrm{CH}^1(X) \cong \mathrm{Pic}\, X,$$

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so these components are torsion-free. We treat the next case in Theorem 3.1 (i), whose proof occupies all of Section 2 and most of Section 3:

Theorem. *Let G be a split simple linear algebraic group over a field k. Then the torsion subgroup of $\gamma^{2/3}$ for the variety of Borel subgroups of G is $\mathbb{Z}/N(G)\mathbb{Z}$, for $N(G)$ the Dynkin index of G.*[1)]

In particular, *if G has type E_8, then $\gamma^{2/3}$ has torsion subgroup $\mathbb{Z}/60\mathbb{Z}$.* The table in Section 1.4 shows that $N(G) = 1$ iff G_i has type A or C, so using Lemma 3.8 we can strengthen Grothendieck's result by observing that if $\oplus\gamma^{i/i+1}$ has torsion, then it can already be seen in $\gamma^{2/3}$:

Corollary. *Let G be a semisimple linear algebraic group over an algebraically closed field k. Then*

$$\oplus_{i\geq 0}\gamma^{i/i+1} \textit{ is torsion-free if and only if } \gamma^{2/3} \textit{ is torsion-free,}$$

if and only if the simply connected cover of G is isomorphic to a product of copies of SL_n and Sp_{2n} for various n.

Recall that the Rost invariant is a map $r_G\colon H^1(k,G) \to H^3(k,\mathbb{Q}/\mathbb{Z}(2))$ that is functorial in k and defined for every simple simply connected algebraic group G. For such G, it is, roughly speaking, the first nonzero invariant [20, §31] and it generates the group of invariants with codomain $H^3(*,\mathbb{Q}/\mathbb{Z}(2))$, see [13]. It plays an important role in the study of quadratic forms (where it is known as the Arason invariant) and it was crucial in Bayer and Parimala's proof of the Hasse Principle Conjecture II for classical groups in [4] and the proof of all known cases of the conjecture for exceptional groups.

The proof of our core result gives a class θ that generates the torsion in $\gamma^{2/3}$ in case G is split simple. In Section 4 we argue that θ is also defined for adjoint G, and that for such G's, the image of θ under the map $\gamma^{2/3} \to \mathrm{CH}^2(X)$ can be viewed as an extension of the Rost invariant to adjoint groups – for this to make sense, some details need to be checked, see Proposition 4.4. Because our generalization looks rather different, we give an illustration (Proposition 4.6) to show how it can be used.

As $\mathrm{CH}^0(X)$ and $\mathrm{CH}^1(X)$ are known, and the torsion in $\mathrm{CH}^2(X)$ is known by the theory of the Rost invariant (see Proposition 3.2), we next consider the torsion in $\mathrm{CH}^3(X)$. We drop our hypothesis that G is split, and instead require merely that G is *strongly inner*.[2)] Little is known about $\mathrm{CH}^3(X)$ in general, not even if it is finitely generated. We use our core result to bound the torsion in $\mathrm{CH}^3(X)$ (see Proposition 5.2). As an example, we have: if G is *versal* (in the language of [13, pp. 11, 12], roughly meaning "generic"), then the torsion part of $\mathrm{CH}^3(X)$ can consist only of subgroups $\mathbb{Z}/2^s\mathbb{Z}$ for $s \leq 4$, $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/5\mathbb{Z}$. In particular, Proposition 5.4 gives:

Example. If G is a versal group of type E_8, then the group of odd-order elements in $\mathrm{CH}^3(X)$ is isomorphic to $(\mathbb{Z}/15\mathbb{Z})^{\oplus 8}$.

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1) The Dynkin index of G is defined in Section 1.4.

2) See Section 1.4 for a precise definition. Standard examples are any split group; any group isogenous to $\mathrm{SO}(q)$ for q a quadratic form with even dimension, trivial discriminant, and split Clifford algebra; and any group of type G_2, F_4, or E_8.

The techniques developed in the paper as well as some of the obtained results have been successfully applied to study more general examples of groups and torsors (not necessary strongly inner). For instance, the case of HSpin-torsors was investigated in [34]. The case of PGO_8^+-torsors was studied in [28] together with applications to the motivic classification of algebras with orthogonal involutions. The paper [18] used the obtained results to describe Tits indices of inner groups of type E_6. In [2] and [3] it was used to study the torsion in CH^i for $i \geq 4$ of twisted Spin-flags.

1. Preliminaries

We now provide several facts and observations concerning Chow groups, characteristic maps, invariants, Dynkin indices and filtrations on K_0 for varieties of Borel subgroups of split simple linear algebraic groups.

1.1. Two filtrations on K_0. All facts provided here can be found in [5], [19, §2], [11, §15] and [12, Chapters 3, 5]. Let X be a smooth projective variety over a field k. Consider the γ-filtration on $K_0(X)$. It is given by the subgroups

$$\gamma^i(X) = \big\langle c_{n_1}^{K_0}(b_1)\cdots c_{n_m}^{K_0}(b_m) \mid n_1+\cdots+n_m \geq i \text{ and } b_1,\ldots,b_m \in K_0(X)\big\rangle,$$

where $c_n^{K_0}$ denote the n-th Chern class with values in K_0. For example, for the class of a line bundle we have $c_1^{K_0}([\mathcal{L}]) = 1-[\mathcal{L}^*]$. Let $\gamma^{i/i+1}(X) = \gamma^i(X)/\gamma^{i+1}(X)$ denote the respective quotient. Consider the topological filtration on $K_0(X)$ given by the subgroups

$$\tau^i(X) = \langle [\mathcal{O}_V] \mid V \hookrightarrow X \text{ and codim } V \geq i\rangle,$$

where $[\mathcal{O}_V]$ is the class of the structure sheaf of a closed subvariety V. Let

$$\tau^{i/i+1}(X) = \tau^i(X)/\tau^{i+1}(X)$$

denote the corresponding quotient.

There is an obvious surjection $\mathrm{pr}\colon \mathrm{CH}^i(X) \twoheadrightarrow \tau^{i/i+1}(X)$ from the Chow group of codimension i cycles given by $V \mapsto [\mathcal{O}_V]$. By the Riemann–Roch Theorem the i-th Chern class induces the map in the opposite direction,

$$c_i\colon \tau^{i/i+1}(X) \to \mathrm{CH}^i(X),$$

and the composite $c_i \circ \mathrm{pr}$ is the multiplication by $(-1)^{i-1}(i-1)!$ which is an isomorphism for $i \leq 2$, see [11, Example 15.3.6]. For example, by the very definition we have

$$c_i\Big(\prod_{j=1}^{i} c_1^{K_0}([\mathcal{L}_j])\Big) = (-1)^{i-1}(i-1)!\prod_{j=1}^{i} c_1^{\mathrm{CH}}(\mathcal{L}_j),$$

where $\mathcal{L}_j$ is a line bundle. Observe also that c_i becomes an isomorphism after tensoring with $\mathbb{Q}$.

There is an inclusion $\gamma^i(X) \subseteq \tau^i(X)$ for all i. Moreover, $\gamma^i(X) = \tau^i(X)$ for $i \leq 2$. Observe that $\gamma^{1/2}(X) = \tau^{1/2}(X) = \mathrm{CH}^1(X)$ is the Picard group and by [19, Corollary 2.15] there is an exact sequence

$$0 \to \tau^3(X)/\gamma^3(X) \to \mathrm{Tors}\,\gamma^{2/3}(X) \xrightarrow{c_2} \mathrm{Tors}\,\mathrm{CH}^2(X) \to 0, \tag{1}$$

where we have written c_2 for the composition $\gamma^{2/3}(X) \to \tau^{2/3}(X) \xrightarrow{c_2} \mathrm{CH}^2(X)$ and Tors for the torsion subgroup.

1.2. Characteristic maps and invariants. Let G_s be a split simply connected simple linear algebraic group of rank n over a field k. We fix a split maximal torus T and a Borel subgroup B such that $T \subset B \subset G_s$. Let $\mathfrak{B}_s$ denote the variety of Borel subgroups of G_s and let T^* denote the group of characters of T. We fix a basis of T^* given by the fundamental weights $\omega_1, \dots, \omega_n$.

Let $\mathbb{S}(T^*)$ be the symmetric algebra of T^*. Its elements are polynomials in the fundamental weights ω_i with coefficients in $\mathbb{Z}$. Let $\mathbb{Z}[T^*]$ be the integral group ring of T^*. Its elements are integral linear combinations $\sum_i a_i e^{\lambda_i}$, $\lambda_i \in T^*$. Consider the characteristic maps for CH and K_0 (see [7, §§8, 9] and [8, §§1.5, 1.6]),

$$\mathrm{c}\colon \mathbb{S}(T^*) \to \mathrm{CH}(\mathfrak{B}_s) \quad \text{and} \quad \mathrm{c}'\colon \mathbb{Z}[T^*] \to K_0(\mathfrak{B}_s)$$

given by

$$\mathrm{c}\colon \omega_i \mapsto c_1^{\mathrm{CH}}(\mathcal{L}(\omega_i)) \quad \text{and} \quad \mathrm{c}'\colon e^{\lambda} \mapsto [\mathcal{L}(\lambda)],$$

where $\mathcal{L}(\lambda)$ is the line bundle over $\mathfrak{B}_s$ associated to the character λ.

There are obvious augmentation maps $\mathbb{S}(T^*) \to \mathbb{Z}$ and $\mathrm{aug}\colon \mathbb{Z}[T^*] \to \mathbb{Z}$ given by $\omega_i \mapsto 0$ and $e^{\lambda} \mapsto 1$ respectively. The Weyl group acts naturally on T^*, hence also on $\mathbb{S}(T^*)$ and $\mathbb{Z}[T^*]$. Consider the subrings of invariants $\mathbb{S}(T^*)^W$ and $\mathbb{Z}[T^*]^W$. We denote $\ker \mathrm{c} = I$ and $\ker \mathrm{c}' = I'$. Therefore we have embeddings

$$\mathrm{c}\colon \mathbb{S}(T^*)/I \hookrightarrow \mathrm{CH}(\mathfrak{B}_s) \quad \text{and} \quad \mathrm{c}'\colon \mathbb{Z}[T^*]/I' \xrightarrow{\simeq} K_0(\mathfrak{B}_s),$$

where the second map is surjective since G_s is simply connected [25].

Observe that the kernel I' is the ideal generated by the elements of $\mathbb{Z}[T^*]^W$ from the kernel of the augmentation map. By [7, §2, Corollary 2] the kernel I of c consists of elements g such that

$$m \cdot g = \sum_i g_i \cdot f_i, \tag{2}$$

for $m \in \mathbb{Z}$, f_i the basic polynomial invariants, and $g_i \in \mathbb{S}(T^*)$.

There is a W-invariant quadratic form q on $T^* \otimes \mathbb{Q}$ that is uniquely determined up to a scalar multiple [6, §§VI.1.1–2]. We normalize q so that it takes the value 1 on every short coroot; as q is indivisible, it can be taken as the generator of I of degree 2. To say it differently, each element of I of degree 2 is a multiple of q by an integer.

The form q should be familiar. Its polar bilinear form b_q amounts to the restriction of the "reduced Killing form" to the Cartan subalgebra of the Lie algebra of G_s as described in [16, §5]. In the case where the roots are all one length, an explicit formula for b_q is well known: its Gram matrix is the Cartan matrix of the root system.

If G_s is not of type A_n ($n \geq 2$), then there is no basic invariant of degree 3 [17, p. 59], so by (2) and the indivisibility of q, every $g \in I$ of degree 3 can be written as $g = (\sum a_i \omega_i) q$ for some $a_i \in \mathbb{Z}$.

1.3. The γ-filtration on the variety of Borel subgroups. Consider the γ-filtration on the variety $\mathfrak{B}_s$ of Borel subgroups of G_s. Let γ^m denote the subgroup of $\mathbb{Z}[T^*]$ generated by products of at least m elements of the form $(1-e^{-\omega_i})$, where ω_i is a fundamental weight. Then the isomorphism c′ induces an isomorphism

$$\gamma^{m/m+1}(\mathfrak{B}_s) \simeq \gamma^m/(\gamma^{m+1} + I') \quad \text{for each } i.$$

For example $\gamma^{1/2}(X) \simeq \gamma^1/(\gamma^2 + I')$ is a free abelian group with a basis given by the classes of the elements

$$(1 - e^{-\omega_i}) \in \gamma^1, \quad i = 1, \dots, n.$$

Indeed, $c_1^{K_0}([\mathcal{L}(\omega_i)]) = 1 - [\mathcal{L}(-\omega_i)]$, the map $c_1\colon \gamma^{1/2}(\mathfrak{B}_s) \to \mathrm{CH}^1(\mathfrak{B}_s)$ is an isomorphism and the elements $c_1(\mathcal{L}(\omega_i))$ for $i = 1, \dots, n$ form a basis of the Picard group $\mathrm{CH}^1(\mathfrak{B}_s)$.

Since $K_0(\mathfrak{B}_s)$ is generated by classes of line bundles [25], so is $\gamma^i(\mathfrak{B}_s)$. Therefore, we have

$$\gamma^i(\mathfrak{B}_s) = \big\langle c_1^{K_0}([\mathcal{L}_1]) \cdots c_1^{K_0}([\mathcal{L}_m]) \mid m \geq i \text{ and } \mathcal{L}_j \text{ is a line bundle over } \mathfrak{B}_s \big\rangle.$$

Let $\lambda = \sum_i a_i \omega_i$ be a presentation of a character λ in terms of the fundamental weights. Then $\mathcal{L}(\lambda) = \otimes_i \mathcal{L}(\omega_i)^{\otimes a_i}$. Since for any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ we have

$$c_1^{K_0}([\mathcal{L}_1 \otimes \mathcal{L}_2]) = c_1^{K_0}([\mathcal{L}_1]) + c_1^{K_0}([\mathcal{L}_2]) - c_1^{K_0}([\mathcal{L}_1]) c_1^{K_0}([\mathcal{L}_2]),$$

applying this formula recursively we can express any element of $\gamma^{i/i+1}(\mathfrak{B}_s)$ as a linear combination of the products of the first Chern classes of the bundles $\mathcal{L}(\omega_i)$, $i = 1, \dots, n$. For instance, any element of $\gamma^{2/3}(\mathfrak{B}_s)$ can be written as a class of

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (1 - e^{-\omega_i})(1 - e^{-\omega_j}) \in \gamma^2 \mod \gamma^3 + I', \quad \text{where } a_{ij} \in \mathbb{Z}.$$

1.4. The Dynkin index. Let N denote the map $\mathbb{Z}[T^*]^W \to \mathbb{Z}$ defined by fixing a long root α and setting

$$N\Big(\sum_i a_i e^{\lambda_i}\Big) := \frac{1}{2} \sum_i a_i \langle \lambda_i, \alpha^\vee \rangle^2$$

as in [13, p. 133]. This does not depend on the choice of α and takes values in $\mathbb{Z}$ (and not merely in $\frac{1}{2}\mathbb{Z}$), cf. Lemma 2.5 below. The number $N(\chi)$ is called the *Dynkin index of* χ. Note that for $m \in \mathbb{Z}$, we have $N(m) = N(me^0) = 0$, so $N(\chi)$ only depends on the image of χ in the kernel of the augmentation map.

In case G_s has two root lengths, it is natural to wonder what one would find if one used a short root, say, δ in the definition of N instead of the long root α. We claim that

$$\frac{1}{2} \sum a_i \langle \lambda_i, \delta^\vee \rangle^2 = q(\delta^\vee) \Big[\frac{1}{2} \sum a_i \langle \lambda_i, \alpha^\vee \rangle^2 \Big], \tag{3}$$

where q is the form introduced in Section 1.2. In other words, one obtains something that differs by a factor of $q(\delta^\vee)$. (We will use this observation later.) To prove it, define quadratic forms n_α and n_δ on T^* via $n_\alpha(\lambda) = \sum_{w \in W} \langle w\lambda, \alpha^\vee \rangle^2$ and similarly for $\delta^\vee$. For example, $n_\delta(\alpha) = q(\delta^\vee)^2 n_\alpha(\delta)$. But n_α is a W-invariant quadratic form on T^*, hence it is a scalar multiple of q. As $q(\alpha) = q(\delta^\vee) q(\delta)$, we have $n_\alpha(\alpha) = q(\delta^\vee) n_\alpha(\delta)$. But n_δ is also a scalar multiple of q, so we conclude that $n_\delta = q(\delta^\vee) n_\alpha$, proving the claim.

The *Dynkin index* $N(G_s)$ is defined to be the gcd of $N(\chi)$ as χ varies over the characters of finite-dimensional representations of G_s. The number $N(G_s)$ is calculated in [13, 21, 22]:

type of G_s	A or C	B_n ($n \geq 3$), D_n ($n \geq 4$), G_2	F_4 or E_6	E_7	E_8
$N(G_s)$	1	2	6	12	60

If G is a simple and strongly inner group, then, for the purposes of this paper, we define the Dynkin index $N(G)$ of G to be the Dynkin index $N(G_s)$ of the split simply connected group of the same Killing–Cartan type.[3)]

2. Dynkin indices and the map ϕ

The purpose of this section is to prove Corollary 2.6, which will play an important role in the proof of Theorem 3.1.

Let G_s denote a split simply connected simple linear algebraic group of rank n over a field k. We fix a pinning for G_s and in particular a split maximal torus T and fundamental weights $\omega_1, \ldots, \omega_n$. As G_s is simply connected, T_* $(= \operatorname{Hom}(\mathbb{G}_m, T))$ and T^* are canonically identified with the coroot and weight lattices respectively.

Definition 2.1. Put $\mathbb{Z}[T^*] := \mathbb{Z}[e^{\omega_1}, \ldots, e^{\omega_n}]$ for the integral group ring, and $\mathbb{S}(T^*) := \mathbb{Z}[\omega_1, \ldots, \omega_n]$ for the symmetric algebra of T^*. We define a ring homomorphism

$$\phi_m \colon \mathbb{Z}[T^*]/\gamma^{m+1} \to \mathbb{S}(T^*)/(\mathbb{S}^{m+1}(T^*)), \quad m \geq 2,$$

via

$$\phi_m\big(e^{\sum_{i=1}^n a_i \omega_i}\big) = \prod_{i=1}^n (1-\omega_i)^{-a_i}.$$

In particular, $\phi_m(e^{\omega_i}) = 1 + \omega_i + \cdots + \omega_i^m$ and $\phi_m(e^{-\omega_i}) = 1 - \omega_i$.[4)]

The homomorphism ϕ_m is an isomorphism. To see this, define a homomorphism $\mathbb{S}(T^*) \to \mathbb{Z}[T^*]/\gamma^{m+1}$ via $\psi_m(\omega_i) = 1 - e^{-\omega_i}$ for all i; it induces a homomorphism $\mathbb{S}(T^*)/(\mathbb{S}^{m+1}(T^*)) \to \mathbb{Z}[T^*]/\gamma^{m+1}$ that we also denote by ψ_m. As the compositions $\phi_m\psi_m$ and $\psi_m\phi_m$ are both the identity on generators, the claim is proved.

Proposition 2.2. *If G_s is simple, then for $\chi \in \mathbb{Z}[T^*]^W$, we have*

$$\phi_2(\chi) = \operatorname{aug}(\chi) + N(\chi) \cdot q \in \big(\mathbb{S}(T^*)/(\mathbb{S}^3(T^*))\big)^W,$$

where q is the invariant form introduced in Section 1.2.

3) Recall that there is a unique split simply connected group G_s of the same Killing–Cartan type as G. Following [32], the group G is said to be *strongly inner* if the simply connected cover of G is isomorphic to G_s twisted by a cocycle $\xi \in H^1(k, G_s)$.

4) Note that $\mathbb{Z}[T^*]$ can be viewed as Laurent polynomials in the variables $\omega_1, \ldots, \omega_n$, and from this perspective it is clear that the formula for ϕ gives a well-defined ring homomorphism on $\mathbb{Z}[T^*]$ and $\phi_m(\gamma^{m+1})$ is zero in $\mathbb{S}(T^*)/(\mathbb{S}^{m+1}(T^*))$.

The proof would be much easier if we already knew that ϕ_2 takes W-invariant elements to W-invariant elements, but this only comes as a consequence of the proof of the proposition. We give some preliminary material before the proof.

Example 2.3 (SL_2). In case $G_s = \mathrm{SL}_2$, write ω for the unique fundamental weight. For $n > 0$, we have

$$\phi_2(e^{n\omega} + e^{-n\omega}) = (1 + \omega + \omega^2)^n + (1 - \omega)^n = 2 + n^2\omega^2,$$

which verifies Proposition 2.2 for this group.

Example 2.4 ($\mathrm{SL}_2 \times \mathrm{SL}_2$). In case $G_s = \mathrm{SL}_2 \times \mathrm{SL}_2$ there are two fundamental weights ω_1, ω_2 and the Weyl group W is the Klein four-group; it acts by flipping the signs of ω_1 and ω_2. The definition of ϕ_2 above makes sense here even though G_s is not simple. We find

$$\phi_2(We^{a_1\omega_1 + a_2\omega_2}) = 4 + 2\big[a_1^2\omega_1^2 + a_2^2\omega_2^2\big].$$

Let us give one final observation about Weyl group actions. We write $W\lambda$ for the W-orbit of $\lambda \in T^*$.

Lemma 2.5. *For every root α and weight $\lambda \in T^*$, the map $W\lambda \to \mathbb{Z}$ defined by $\pi \mapsto \langle \pi, \alpha^\vee \rangle$ hits x and $-x$ the same number of times, for every $x \in \mathbb{Z}$. If α, β are orthogonal roots, then for every weight $\lambda \in T^*$, the map $W\lambda \to \mathbb{Z} \times \mathbb{Z}$ defined by $\pi \mapsto (\langle \pi, \alpha^\vee \rangle, \langle \pi, \beta^\vee \rangle)$ hits (x, y), $(-x, y)$, $(x, -y)$, and $(-x, -y)$ the same number of times, for every $x, y \in \mathbb{Z}$.*

Proof. We prove the second claim. The reflections s_α, s_β generate a copy V of the Klein four-group in W. Fix a decomposition of W into V-cosets: $W = \bigcup V w_i$. For each $v \in V$, we have

$$\big(\langle v w_i \lambda, \alpha^\vee \rangle, \langle v w_i \lambda, \beta^\vee \rangle\big) = \big(\langle w_i \lambda, v\alpha^\vee \rangle, \langle w_i \lambda, v\beta^\vee \rangle\big) \in \mathbb{Z} \times \mathbb{Z}.$$

If we write (x, y) for the value obtained at $v = 1$, then the other three elements of V give $(-x, y)$, $(x, -y)$, and $(-x, -y)$. Therefore, the map $f\colon W \mapsto \mathbb{Z} \times \mathbb{Z}$ defined by

$$f(w) = \big(\langle w\lambda, \alpha^\vee \rangle, \langle w\lambda, \beta^\vee \rangle\big)$$

produces these four elements the same number of times, for each $x, y \in \mathbb{Z}$. The claim follows, because it concerns the map obtained by dividing f by the order of the stabilizer of λ in W. □

Proof of Proposition 2.2. We may assume that $\chi = \sum e^{\lambda_j}$ where $\lambda_1, \dots, \lambda_r$ is the Weyl orbit of some $\lambda \in T^*$. Put $\lambda_j = \sum_{i=1}^n a_{ij}\omega_i$, so $\phi(\chi) = \sum_{j=1}^r \prod_{i=1}^n (1 - \omega_i)^{-a_{ij}}$. Obviously, the degree 0 component of $\phi(\chi)$ is $r = \mathrm{aug}(\chi)$.

The degree 1 component of $\phi(\chi)$ is $\sum_j \sum_i a_{ij}\omega_i = \sum_i (\sum_j a_{ij})\omega_i$. Here the claim is that $\sum_j a_{ij} = 0$ for each i. The a_{ij}'s are the images of $W\lambda$ in $\mathbb{Z}$ under the map $\lambda_j \mapsto \langle \lambda_j, \alpha_i^\vee \rangle$ where α_i denotes the simple root corresponding to the fundamental weight ω_i, hence the claim follows from Lemma 2.5.

The crux is to check the claim on the degree 2 component q_1 of $\phi(\chi)$; it is an integer-valued quadratic form on the coroot lattice T_* and we check that it equals $q_2 := N(\chi)q$. We write out for $\ell = 1, 2$:

$$q_\ell\Big(\sum d_i\alpha_i^\vee\Big) = \sum_i d_i^2 q_\ell(\alpha_i^\vee) + \sum_{i<j} d_i d_j b_{q_\ell}(\alpha_i^\vee, \alpha_j^\vee), \tag{4}$$

where b_{q_ℓ} is the polar bilinear form of q_ℓ. We will check that the value of this expression is the same for $\ell = 1, 2$.

First suppose that $\delta^\vee := \sum d_i \alpha_i^\vee$ is a coroot and every d_i is 0 or 1. Then it defines a homomorphism $\eta\colon \mathrm{SL}_2 \to G_s$ so that, roughly speaking, the simple coroot $\alpha^\vee$ of SL_2 (viewed as a map $\mathbb{G}_m \to T_1 := \eta^{-1}(T)$) satisfies $\eta(\alpha^\vee) = \delta^\vee$. We check that the diagram

$$\begin{tikzcd} \mathbb{Z}[T^*] \arrow[r, "\phi_2"] \arrow[d, "\eta^*"'] & \mathbb{S}(T^*)/(\mathbb{S}^3(T^*)) \arrow[d, "\eta^*"] \\ \mathbb{Z}[T_1^*] \arrow[r, "\phi_2"] & \mathbb{S}(T_1^*)/(\mathbb{S}^3(T_1^*)) \end{tikzcd} \tag{5}$$

commutes. Since $\omega_j(\delta^\vee) = d_j$, we have $\eta^*(\omega_j) = d_j\omega$ for ω the fundamental weight of SL_2 dual to $\alpha^\vee$. We find

$$\eta^*\phi_2\big(e^{\sum c_j\omega_j}\big) = \prod_j (1 - d_j\omega)^{-c_j} = (1-\omega)^{-\sum c_j d_j},$$

because the d_j's are all 0 or 1. As this is $\phi_2(e^{(\sum d_j c_j)\omega}) = \phi_2\eta^*(e^{\sum c_j\omega_j})$, we have confirmed the commutativity of (5).

Put ϕ^2 for the composition of ϕ_2 with the projection onto the degree 2 component $\mathbb{S}^2$, so $q_1 = \phi^2(\chi)$. Then $q_1(\delta^\vee) = (\eta^*\phi^2(\chi))(\alpha^\vee)$ obviously, which is $(\phi^2\eta^*(\chi))(\delta^\vee)$ by commutativity of (5). We have $\eta^*(\chi) = \sum_j e^{\sum_i a_{ij}\omega}$ and by Lemma 2.5, the multiset of the j integers $\sum_i a_{ij}d_i$ is symmetric under multiplication by -1, hence by Example 2.3 we find

$$q_1(\delta^\vee) = \frac{1}{2}\Big(\sum_j\Big(\sum_i a_{ij}d_i\Big)^2\Big) = \frac{1}{2}\sum_j \langle\lambda_j, \delta^\vee\rangle^2.$$

By (3) this equals $q(\delta^\vee)N(\chi) = q_2(\delta^\vee)$.

Returning to equation (4), this shows that the term $q_\ell(\alpha_i^\vee)$ does not depend on ℓ. Similarly, if $\alpha_i^\vee$ and $\alpha_j^\vee$ are not orthogonal coroots, then $\alpha_i^\vee$ and $\alpha_j^\vee$ are adjacent in the Dynkin diagram and $\alpha_i^\vee + \alpha_j^\vee$ is a coroot [6, VI.1.6, Corollary 3b]. The preceding two paragraphs show that the value of

$$b_{q_\ell}(\alpha_i^\vee, \alpha_j^\vee) = q_\ell(\alpha_i^\vee + \alpha_j^\vee) - q_\ell(\alpha_i^\vee) - q_\ell(\alpha_j^\vee)$$

does not depend on ℓ.

It remains to consider $b_{q_\ell}(\alpha_i^\vee, \alpha_j^\vee)$ where $\alpha_i^\vee$ and $\alpha_j^\vee$ are orthogonal (relative to the polar form of q; it follows that they are orthogonal relative to b_{q_2}. We use $\alpha_i^\vee$ and $\alpha_j^\vee$ to define a homomorphism $\tau\colon \mathrm{SL}_2 \times \mathrm{SL}_2 \to G_s$ and – as we did for SL_2 above – we fix a torus $T_2 = T_1 \times T_1 \subset \mathrm{SL}_2 \times \mathrm{SL}_2$ such that $\tau(T_2) = \mathrm{im}(\alpha_i^\vee \times \alpha_j^\vee) \subset T$. Arguing using a commutative diagram analogous to (5), it suffices to check that the simple roots of $\mathrm{SL}_2 \times \mathrm{SL}_2$ are orthogonal relative to $\tau q_1 = \phi^2\tau(\chi)$, which follows from Example 2.4 and Lemma 2.5. □

In view of Section 1.4, Proposition 2.2 gives:

Corollary 2.6. $\phi_2(I') = \mathbb{Z}\cdot N(G_s)\cdot q$. □

3. Torsion in the γ-filtration

Let $\mathfrak{B}$ denote the variety of Borel subgroups of a strongly inner simple linear algebraic group G over k. Observe that the variety $\mathfrak{B}$ is always defined over k by [9, Corollary XXVI.3.6]; it is a twisted form of the variety of Borel subgroups $\mathfrak{B}_s$ of G_s, i.e., $\mathfrak{B}$ and $\mathfrak{B}_s$ become isomorphic over the algebraic closure of k.

In the present section we determine and bound respectively the torsion parts of the second and the third quotients of the γ-filtration on the variety $\mathfrak{B}$. The main result is the following.

Theorem 3.1. *Let $\mathfrak{B}$ be the variety of Borel subgroups of a strongly inner simple linear algebraic group G over a field k. Then:*

(i) $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$ *is a cyclic group of order the Dynkin index $N(G)$ and generated by $\mathrm{c}'(\theta)$ for θ as in Definition* 3.4.

(ii) *The subgroup $\tau^3(\mathfrak{B})/\gamma^3(\mathfrak{B})$ of* $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$ *is generated by $o(r(G))\,\mathrm{c}'(\theta)$.*

(iii) $2\operatorname{Tors}\gamma^{3/4}(\mathfrak{B})$ *is a quotient of $(\mathbb{Z}/N(G))^{\oplus(\operatorname{rank}G)}$.*

The element $r(G)$ in (ii) is defined as follows. Put G_s for the simply connected split group of the same Killing–Cartan type as G. By definition, the simply connected cover of G is isomorphic to G_s twisted by a class $\xi \in H^1(k, G_s)$. The image $r_{G_s}(\xi) \in H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ of ξ under the Rost invariant depends only on the isomorphism class of G [14, Lemma 2.1], and so we denote it simply by $r(G)$.[5)] We put $o(r(G))$ for its order in the abelian group $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$; it divides $N(G)$ by [13, pp. 133, 135].

Philippe Gille pointed out to us at the beginning of this project that pasting together two results in the literature gives a description of $\operatorname{Tors}\mathrm{CH}^2(X)$ for some X.

Proposition 3.2. *Let X be a projective homogeneous variety under G. If G is split by $k(X)$, then* $\operatorname{Tors}\mathrm{CH}^2(X)$ *is a cyclic group whose order is the same as the order of $r(G)$ in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$; in particular its order divides $N(G)$.*

Proof. We view ξ as a principal homogeneous G_s-variety. The kernel of the scalar extension map $H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(\xi), \mathbb{Q}/\mathbb{Z}(2))$ is the cyclic group generated by $r(G)$ by [13, p. 129]. For every extension L/k, ξ has a point over L if and only if G is split, if and only if X has a point over L. Therefore, this kernel is the same as the kernel of $H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(X), \mathbb{Q}/\mathbb{Z}(2))$. A theorem of Peyre–Merkurjev [27] shows that this kernel is isomorphic to $\operatorname{Tors}\mathrm{CH}^2(X)$. □

Remark 3.3. Obviously, one can take $X = \mathfrak{B}$ in the proposition. Furthermore, the same proof shows that the proposition still holds if one replaces "G is strongly inner" and "G is split by $k(X)$" with "G has trivial Tits algebras" and "G becomes quasi-split over $k(X)$".

Also, the statement in Theorem 3.1 (i) makes use of the following definition.

5) Recall that $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ is the direct sum over primes p of the groups $H^3(k, \mathbb{Q}_p/\mathbb{Z}_p(2))$, defined as Galois cohomology for $p \neq \operatorname{char} k$ and using K-theory for $p = \operatorname{char} k$, see [13, pp. 151, 152].

Definition 3.4. Write the form q from Section 1.2 (relative to the split group G_s) as $q = \sum_{i \leq j} c_{ij}\omega_i\omega_j \in \mathbb{S}(T^*)$. We call the element

$$\theta := \sum_{i \leq j} c_{ij}(1 - e^{-\omega_i})(1 - e^{-\omega_j}) \in \mathbb{Z}[T^*]$$

the *special cycle*. Its image in $\mathbb{Z}[T^*]/\gamma^{m+1}$ is $\psi_m(q)$ for all $m \geq 2$. Applying c' gives an element of $\gamma^2(\mathfrak{B}_s)$.

By the result of Panin [25, Theorem 2.2.(2)] since G is strongly inner, the restriction map

$$\text{(6)} \qquad \mathrm{res}\colon K_0(\mathfrak{B}) \to K_0(\mathfrak{B} \times k_{\mathrm{alg}}) \simeq K_0(\mathfrak{B}_s \times k_{\mathrm{alg}}) \simeq K_0(\mathfrak{B}_s)$$

is an isomorphism, where k_{alg} denotes an algebraic closure of k. Since the γ-filtration is defined in terms of Chern classes and the latter commute with restrictions, it induces an isomorphism between the γ-quotients, i.e.,

$$\mathrm{res}\colon \gamma^{i/i+1}(\mathfrak{B}) \xrightarrow{\simeq} \gamma^{i/i+1}(\mathfrak{B}_s).$$

The inverse image of $\mathrm{c}'(\theta)$ under this isomorphism belongs to $\gamma^{2/3}(\mathfrak{B})$ and we abuse notation by also denoting it by $\mathrm{c}'(\theta)$.

Proof of Theorem 3.1. Via the previous paragraph, we may assume that $G = G_s$. Let T, T^*, etc. be as in Section 1.2.

There is a commutative diagram

$$\text{(7)} \qquad \begin{array}{ccc} \gamma^{m/m+1}(\mathfrak{B}_s) & \xrightarrow{\quad c_m \quad} & \mathrm{CH}^m(\mathfrak{B}_s) \\ {\scriptstyle \mathrm{c}'}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{c}} \\ \gamma^m/\gamma^{m+1} & \xrightarrow{(-1)^{m-1}(m-1)!\cdot\phi_m} & \mathbb{S}^{\geq m}(T^*)/\,\mathbb{S}^{>m}(T^*). \end{array}$$

First take $m = 2$ and suppose that $x \in \gamma^{2/3}$ maps to a torsion element in $\gamma^{2/3}(\mathfrak{B}_s)$. As $\mathrm{CH}^2(\mathfrak{B}_s)$ has zero torsion, the commutativity of (7) shows that $\phi_2(x)$ is in the kernel I of c. Writing $x = \sum_{i,j} a_{ij}(1 - e^{\omega_i})(1 - e^{\omega_j}) \bmod \gamma^3$, we have $\phi_2(x) = \sum a_{ij}\omega_i\omega_j$ of degree 2 in I, hence $\phi_2(x) = aq$ for some $a \in \mathbb{Z}$. Then modulo γ^3, we have $x \equiv \psi_2\phi_2(x) \equiv a\theta$, so $\mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B}_s)$ is a cyclic group generated by the class of the special cycle θ modulo $\gamma^3 + I'$.

By Corollary 2.6 there exists $\chi \in I'$ such that $\phi_2(\chi) = N(G_s) \cdot q$. Applying ψ_2 we obtain that

$$0 \equiv \chi \equiv N(G_s) \cdot \theta \mod \gamma^3 + I',$$

hence, the order of θ modulo $\gamma^3 + I'$ divides the Dynkin index $N(G_s)$. This shows that $\mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B})$ is a cyclic group of order dividing $N(G)$ with generator $\mathrm{c}'(\theta)$.

Let $\xi' \in H^1(k', G_s)$ be a versal G_s-torsor for some extension k' of k, and write $\mathfrak{B}'$ for the Borel variety (over k') of the group G_s twisted by ξ'. The element $r(\xi')$ has order $N(G_s)$ in $H^3(k', \mathbb{Q}/\mathbb{Z}(2))$ by [13, pp. 31, 133]. But $\mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B}')$ is cyclic of order dividing $N(G_s)$, hence Proposition 3.2 and the exactness of sequence (1) give that $\mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B}')$ also has order $N(G_s)$. Now take K to be an algebraically closed field containing k'. The restriction maps for $k \to K$ and $k' \to K$ give isomorphisms

$$\mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B}_k) \simeq \mathrm{Tors}\,\gamma^{2/3}((\mathfrak{B}_s)_K) \simeq \mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B}'_{k'}),$$

which is itself $\mathbb{Z}/N(G)$, completing the proof of (i). Claim (ii) follows from the exactness of sequence (1).

Now take $m = 3$ and suppose that $x \in \gamma^{3/4}$ maps to a torsion element in $\gamma^{3/4}(\mathfrak{B}_s)$. As $\mathrm{CH}^3(\mathfrak{B}_s)$ has zero torsion, diagram (7) shows that $2\phi_3(x)$ is in the kernel I of c. As in the $m = 2$ case, $2\phi_3(x)$ has degree 3.

Suppose G_s is not of type A_n for $n \geq 2$. Then by Section 1.2, $2\phi_3(x) = q \cdot f$, where $f = \sum_{i=1}^n a_i \omega_i$. Applying ψ_3 we obtain that $2x = \theta \cdot f'$, where $f' = \sum_{i=1}^n a_i(1 - e^{-\omega_i})$. In other words, the torsion part of $2\gamma^{3/4}(\mathfrak{B}_s)$ is generated by the elements $\mathrm{c}'(\theta \cdot (1 - e^{-\omega_i}))$ for $i = 1, \dots, n$.

By Corollary 2.6 there exists $\chi \in I'$ such that $\phi_3(\chi \cdot (1 - e^{-\omega_i})) \equiv N(G_s) \cdot q \cdot \omega_i$ mod $(\mathbb{S}^4(T^*))$. Applying ψ_3 we obtain that

$$0 \equiv \chi \cdot (1 - e^{-\omega_i}) \equiv N(G_s) \cdot \theta \cdot (1 - e^{-\omega_i}) \quad \text{mod } \gamma^4 + I',$$

hence, the torsion part of $2\gamma^{3/4}(\mathfrak{B}_s)$ is a product of n cyclic groups of orders dividing the $N(G_s)$.

One can treat the case where $m = 3$ and G_s is of type A_n for $n \geq 2$ by a small extension of our arguments, but in the interest of space we simply refer to [5, Exposé XIV, §4.5] where it is shown that $\gamma^{i/i+1}(\mathfrak{B}_s)$ is torsion-free, completing the proof of (iii). □

Recall that $\gamma^i(\mathfrak{B}) = \tau^i(\mathfrak{B})$ for $i = 0, 1, 2$. Theorem 3.1 (ii) gives:

Corollary 3.5. $\gamma^3(\mathfrak{B}) = \tau^3(\mathfrak{B})$ *if and only if* $o(r(G)) = N(G)$. □

Remark 3.6. The right value of the order of the Rost invariant for versal torsors is an important unpublished result by M. Rost and occurred first in [10, Appendix C].

Remark 3.7. Note that the order of $r(G)$ is comparatively easy to calculate, in the sense that it suffices to determine the order of the restriction of $r(G)$ over finite extensions $L_1, \dots, L_s$ such that $\gcd\{N(G), L_1, \dots, L_s\} = 1$; in that case, $o(r(G)) = \mathrm{lcm}\{o(r(G \times L_i))\}$.

The next lemma can be used to extend the results obtained in Theorem 3.1 to the case of a semisimple group.

Lemma 3.8. *Let* $G_1, \dots, G_m$, $m > 1$, *be simple and strongly inner groups and write* $\mathfrak{B}_j$ *for the Borel variety of* G_j. *The Borel variety for* $\prod G_j$ *is* $\prod \mathfrak{B}_j$, *and we have*

$$\operatorname{Tors} \gamma^{2/3}\Big(\prod \mathfrak{B}_j\Big) \simeq \bigoplus \operatorname{Tors} \gamma^{2/3}(\mathfrak{B}_j)$$

and

$$\operatorname{Tors} \gamma^{3/4}\Big(\prod \mathfrak{B}_j\Big) \simeq \bigoplus_{j=1}^{m} \big(\operatorname{Tors} \gamma^{3/4}(\mathfrak{B}_j) \oplus \operatorname{Tors} \gamma^{2/3}(\mathfrak{B}_j)\big).$$

Proof. Apply the Künneth decomposition and the fact that $\gamma^{i/i+1}(\mathfrak{B}_j)$ has no torsion for $i = 0$ and 1. □

4. Applications to the Rost invariant

So far, we have studied the case where G is strongly inner and we constructed the special cycle $\mathrm{c}'(\theta)$ in $K_0(\mathfrak{B})$, cf. Example 4.2 below. We prove that under weaker hypotheses on G, the class $\mathrm{c}'(\theta)$ is still defined over k, and we use this result to extend the Rost invariant.

In the present section G_s denotes an adjoint split simple linear algebraic group over a field k. As it is adjoint, the character group T^* of a split maximal torus of G_s is naturally identified with the root lattice Λ_r.

We fix a pinning for G_s, which includes a set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ in Λ_r. Write ω_i for the fundamental weight corresponding to α_i and s_i for the reflection of the weight lattice Λ in the hyperplane orthogonal to α.

4.1. The Steinberg basis. For each element w of the Weyl group W of T we define the weight $\rho_w := \sum_{\{i\in\{1,\dots,n\} \mid w^{-1}(\alpha_i)<0\}} w^{-1}(\omega_i)$. Let $\mathbb{Z}[\Lambda]^W$ denote the subring of W-invariant elements. Then the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^{\rho_w} \mid w \in W\}$ by [30, Theorem 2.2].

Example 4.1. (a) For a simple reflection s_j we have

$$\rho_{s_j} = \sum_{\{i\in\{1,\dots,n\} \mid s_j(\alpha_i)<0\}} s_j(\omega_i) = s_j(\omega_j) = \omega_j - \alpha_j.$$

(b) More generally, let $w := s_{i_1} s_{i_2} \dots s_{i_m}$ be a product of m distinct simple reflections such that the simple roots $\alpha_{i_j}, \alpha_{i_\ell}$ are orthogonal for all $j \neq \ell$. Then

$$\rho_{s_{i_1} s_{i_2} \dots s_{i_m}} = \rho_{s_{i_1}} + \rho_{s_{i_2}} + \dots + \rho_{s_{i_m}}$$

because $w^{-1}(\alpha_i)$ is negative if and only if $i = i_j$ for some j.

(c) For a product of two simple reflections $s_i s_j$ such that $c_{ij} = \alpha_i^\vee(\alpha_j) < 0$ we obtain

$$\rho_{s_i s_j} = \rho_{s_i} + c_{ij}\alpha_j.$$

Let $\mathfrak{B}_s$ denote the variety of Borel subgroups of G_s. Consider the characteristic map $\mathrm{c}' \colon \mathbb{Z}[\Lambda] \twoheadrightarrow K_0(\mathfrak{B}_s)$ for the simply connected cover of G_s. Since the kernel of the surjection c' is generated by elements $x \in \mathbb{Z}[\Lambda]^W$ in the kernel of the augmentation map, there is an isomorphism

$$\mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} \simeq \mathbb{Z}[\Lambda]/\ker(\mathrm{c}') \simeq K_0(\mathfrak{B}_s).$$

The elements

$$\{g_w := \mathrm{c}'(e^{\rho_w}) = [\mathcal{L}(\rho_w)] \mid w \in W\}$$

form a free $\mathbb{Z}$-basis of $K_0(\mathfrak{B}_s)$ called the *Steinberg basis*.

Observe that the quotient group Λ/Λ_r coincides with the group of characters of the center of the simply connected cover of G_s. Consider the surjective ring homomorphism induced by the restriction $\mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/\Lambda_r]$. Since W acts trivially on Λ/Λ_r, we obtain that $\bar{\rho}_w = \sum_{\{i\in\{1,\dots,n\} \mid w^{-1}(\alpha_i)<0\}} \bar{\omega}_i$, where $\bar{\rho}$ means the restriction to Λ/Λ_r.

4.2. The Tits algebras and base change. Let G be a twisted form of G_s, i.e., G is obtained by twisting G_s by a cocycle $\xi \in Z^1(k, \mathrm{Aut}(G_s))$. More specifically, our choice of pinning for G_s defines a section s of the quotient map $\pi \colon \mathrm{Aut}(G_s) \to \mathrm{Aut}(\Delta)$. Twisting G_s by

$\xi' := s\pi(\xi)$ gives a quasi-split group G_q and we pick $\xi'' \in Z^1(k, G_q)$ that maps via twisting to ξ; i.e., we pick ξ'' so that G is isomorphic to ${}_{\xi''}G_q$.

Let $\mathfrak{B} = {}_{\xi}\mathfrak{B}_s$ be the variety of Borel subgroups of G. Let Γ denote the absolute Galois group of k; it acts on the weight lattice Λ via the cocycle ξ'.

Following [31] (see also [25, §§3.1, 11.7] and [23, §2]) we associate with each $\chi \in \Lambda/\Lambda_r$ the field of definition k_χ of χ and the central simple algebra $A_{\chi,\xi}$ over k_χ called the Tits algebra. Here k_χ is a fixed subfield for the stabilizer $\Gamma_\chi = \{\tau \in \Gamma \mid \tau(\chi) = \chi\}$. There is a group homomorphism

$$\beta\colon (\Lambda/\Lambda_r)^{\Gamma_\chi} \to \mathrm{Br}(k_\chi) \text{ with } \beta(\chi') = [A_{\chi',\xi}].$$

4.3. An extension of the Rost invariant. By [27, Theorem 2.1] there is an isomorphism

$$\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}) \simeq \frac{\ker\big(H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(\mathfrak{B}), \mathbb{Q}/\mathbb{Z}(2))\big)}{\bigoplus_{\chi\in\Lambda/\Lambda_r} N_{k_\chi/k}(k_\chi^* \cup \beta(\chi))},$$

where the numerator is the kernel of the restriction map to the field of fractions $k(\mathfrak{B})$ of $\mathfrak{B}$ and $N_{k_\chi/k}$ is the norm map. Let $H^3_\beta(k, \mathbb{Q}/\mathbb{Z}(2))$ denote the cohomology quotient

$$H^3_\beta(k, \mathbb{Q}/\mathbb{Z}(2)) = H^3(k, \mathbb{Q}/\mathbb{Z}(2))/ \bigoplus_{\chi\in\Lambda/\Lambda_r} N_{k_\chi/k}(k_\chi^* \cup \beta(\chi))$$

so that $\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}) \subseteq H^3_\beta(k, \mathbb{Q}/\mathbb{Z}(2))$.

Let l/k be a field extension. Since the Chern classes commute with restrictions, there is the induced map $\mathrm{res}_{l/k}\colon \gamma^{i/i+1}(\mathfrak{B}) \to \gamma^{i/i+1}(\mathfrak{B}_l)$, where $\mathfrak{B}_l = \mathfrak{B} \times_k l$, with the image generated by the products

$$\big\langle c^{K_0}_{n_1}(x_1)\cdots c^{K_0}_{n_m}(x_m) \mid n_1 + \cdots + n_m = i,\ x_1, \ldots, x_m \in \mathrm{res}_{l/k}(K_0(\mathfrak{B}))\big\rangle$$

and there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Tors}\gamma^{2/3}(\mathfrak{B}) \arrow[r, two heads, "c_2"] \arrow[d, "\mathrm{res}_{l/k}"'] & \operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}) \arrow[r, hook] \arrow[d, "\mathrm{res}_{l/k}"] & H^3_\beta(k, \mathbb{Q}/\mathbb{Z}(2)) \arrow[d, "\mathrm{res}_{l/k}"] \\
\operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_l) \arrow[r, two heads, "c_2"] & \operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}_l) \arrow[r, hook] & H^3_\beta(l, \mathbb{Q}/\mathbb{Z}(2)).
\end{tikzcd} \tag{8}$$

Using (8) one can provide a non-trivial element in $H^3_\beta(k, \mathbb{Q}/\mathbb{Z}(2))$ as follows:

- Assume that we are given a non-trivial element over l, i.e., there is a $\theta \in \operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_l)$ such that $c_2(\theta) \in H^3_\beta(l, \mathbb{Q}/\mathbb{Z}(2))$ is non-zero.
- Assume that we know that θ is defined over k, i.e., that $\theta = \mathrm{res}_{l/k}(\theta')$ for some $\theta' \in \operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$.

Then the image $c_2(\theta')$ provides a non-trivial element in $H^3_\beta(k, \mathbb{Q}/\mathbb{Z}(2))$.

Example 4.2. If G is strongly inner – i.e., if G is inner and β is the trivial homomorphism – then for any field extension l/k the left vertical arrow in (8) is an isomorphism, hence, identifying $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$ with the cyclic group generated by the special cycle θ. As in Proposition 3.2 and its proof $\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B})$ coincides with the usual unramified cohomology generated by the Rost invariant $r(G)$ of G and $\langle c_2(\theta)\rangle = \langle r(G)\rangle$ in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$.

Lemma 4.3. *Assume that G is inner.*

(a) *If a weight ω is such that $\beta(\omega) = 0$, then $[\mathcal{L}(\omega)]$ is in the image of*

$$\mathrm{res}\colon K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s).$$

In particular, it holds for the classes $[\mathcal{L}(\alpha_i)]$ of simple roots α_i.

Under the notation of Example 4.1 (b) *we have*

(b) $\sum_j c_1^{K_0}([\mathcal{L}(\omega_{i_j})]) - c_1^{K_0}([\mathcal{L}(\alpha_{i_j})]) \equiv c_1^{K_0}\big(\prod_j g_{s_{i_j}}\big) \equiv c_1^{K_0}(g_w) \mod \gamma^3(\mathfrak{B}_s)$*;*

(c) *if $\beta(\sum_j \omega_{i_j}) = 0$, then $\sum_j c_1^{K_0}([\mathcal{L}(\omega_{i_j})])$ is in the image of*

$$\mathrm{res}\colon \gamma^{1/2}(\mathfrak{B}) \to \gamma^{1/2}(\mathfrak{B}_s).$$

Proof. (a) follows by [15, Corollary 3.1]. (b) follows by the formula for the first Chern class (in K_0) of the tensor product of line bundles. According to (a) each $c_1^{K_0}([\mathcal{L}(\alpha_{i_j})])$ is in the image of the restriction map which implies (c). □

The following proposition says that the Rost invariant can be extended in the prime index case:

Proposition 4.4. *Assume that G is inner and every Tits algebra of ${}_\xi G_s$ has index 1 or p, where p is a prime. Then the special cycle θ is in the image of the restriction map* $\mathrm{res}\colon \gamma^{2/3}(\mathfrak{B}) \to \gamma^{2/3}(\mathfrak{B}_s)$*. In other words, if l/k is an extension that kills* $\mathrm{im}\,\beta$*, then the image of c_2 over l coincides with the subgroup generated by the respective Rost invariant, i.e., we have*

$$\mathrm{im}(c_2)_l = \langle r(G_l)\rangle \subseteq H^3(l, \mathbb{Q}/\mathbb{Z}(2)).$$

Proof. We may assume that $N(G)$ is not 1 (otherwise θ maps to zero in $\gamma^{2/3}(\mathfrak{B}_s)$ by Theorem 3.1) and $\Lambda/\Lambda_r \neq 0$ (otherwise Example 4.2 applies), i.e., we may assume that G has type B, D, E_7 (for $p = 2$) or E_6 (for $p = 3$).

We first make a general observation. Observe that the image $\mathrm{res}_{l/k}(K_0(\mathfrak{B}))$ was determined by Panin in [25]. Since G is inner, Γ acts trivially on Λ/Λ_r, i.e., $k_\chi = k$ for all χ and by [25, Theorem 4.2] the image of the restriction map $K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s)$ from (6) coincides with the sublattice

$$\langle \mathrm{ind}(A_{\bar\rho_w,\xi}) \cdot g_w \mid w \in W\rangle.$$

Modulo $\gamma^3(\mathfrak{B}_s)$, we have

$$\begin{aligned} c_1^{K_0}([\mathcal{L}(\omega_i)])^2 &\equiv \big(c_1^{K_0}(g_{s_i}) + c_1^{K_0}([\mathcal{L}(\alpha_i)])\big)^2 \\ &\equiv c_1^{K_0}(g_{s_i})^2 + 2c_1^{K_0}(g_{s_i})c_1^{K_0}([\mathcal{L}(\alpha_i)]) + c_1^{K_0}([\mathcal{L}(\alpha_i)])^2. \end{aligned}$$

The Whitney Sum Formula gives that $c_2^{K_0}(2g_{s_i}) = c_1^{K_0}(g_{s_i})^2$ and $c_1^{K_0}(2g_{s_i}) \equiv 2c_1^{K_0}(g_{s_i})$ mod $\gamma^2(\mathfrak{B}_s)$. Our hypothesis on the Tits algebras (for $p = 2$) gives that $2g_{s_i}$ is in the image of $K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s)$, and it follows that $c_1^{K_0}([\mathcal{L}(\omega_i)])^2$ is *rational* – i.e., is in the image of $\gamma^{2/3}(\mathfrak{B}) \to \gamma^{2/3}(\mathfrak{B}_s)$ – for all i.

Type E_7. Suppose that G has type E_7. Then

$$q = \Big(\sum_{i=1}^{7} \omega_i^2\Big) - \omega_1\omega_3 - \omega_3\omega_4 - \omega_4\omega_2 - \omega_4\omega_5 - \omega_5\omega_6 - \omega_6\omega_7,$$

where we have numbered the roots following [6]. Each ω_i^2 contributes a term of the form $c_1^{K_0}([\mathcal{L}(\omega_i)])^2$ to the image of $\mathrm{c}'(\theta)$ in $\gamma^{2/3}(\mathfrak{B}_s)$, and such a term is rational by the preceding paragraph. The weights $\omega_1, \omega_3, \omega_4, \omega_6$ belong to the root lattice and so the term $\omega_1\omega_3$ contributes a rational term $c_1^{K_0}([\mathcal{L}(\omega_1)])c_1^{K_0}([\mathcal{L}(\omega_3)])$ to $\mathrm{c}'(\theta)$, and similarly for the term $\omega_3\omega_4$. Next we observe that $\omega_4\omega_2 + \omega_4\omega_5$ contributes

$$c_1^{K_0}([\mathcal{L}(\omega_4)])\big(c_1^{K_0}([\mathcal{L}(\omega_2)]) + c_1^{K_0}([\mathcal{L}(\omega_5)])\big)$$

to $\mathrm{c}'(\theta)$. But ω_4 and $\omega_2+\omega_5$ both lie in the root lattice, so both terms in the product are rational by Lemma 4.3. The same argument handles $\omega_5\omega_6 + \omega_6\omega_7$, and we are done with the E_7 case.

Type D. Suppose that G has type D_n. Then

$$q = \sum_{i=1}^{n} \omega_i^2 - \sum_{i=1}^{n-2} \omega_i\omega_{i+1} - \omega_{n-2}\omega_n.$$

The terms ω_i^2 are treated as in the E_7 case. For the terms in the second sum, we collect around terms with even subscripts: for even $i < n-2$, consider $\omega_i(\omega_{i-1} + \omega_{i+1})$. As ω_i and $\omega_{i-1} + \omega_{i+1}$ belong to the root lattice, we see as in the E_7 case that they contribute rational terms to $\mathrm{c}'(\theta)$.

Suppose now that n is even. Then we have not accounted for $\omega_{n-2}(\omega_{n-3} + \omega_{n-1} + \omega_n)$ from q. As both terms in the product belong to the root lattice, we are finished as in the E_7 case.

If n is odd, then we have not accounted for $\omega_{n-2}(\omega_{n-1} + \omega_n)$ in q. Here Λ/Λ_r is isomorphic to $\mathbb{Z}/4$ and $\omega_{n-2}, \omega_{n-1}, \omega_n$ map to $2, \pm 1, \pm 3$ respectively. In particular, we have $\beta(\omega_{n-2}) = 2\beta(\omega_n)$, which is zero by our hypothesis on the Tits algebras, so $[\mathcal{L}(\omega_{n-2})]$ is in the image of res: $K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s)$. Similarly, $\beta(\omega_{n-1} + \omega_n) = \beta(\omega_{n-1}) + \beta(\omega_n) = 0$, and as in the E_7 case, we see that $\mathrm{c}'(\theta)$ is rational.

Type B. For G of type B_n, Λ/Λ_r equals $\mathbb{Z}/2$ and

$$q = \sum_{i=1}^{n} c_{ii}\omega_i^2 - \sum_{i=1}^{n-1} 2\omega_i\omega_{i+1},$$

where the c_{ii} are 1 or 2. The map $\Lambda \to \Lambda/\Lambda_r$ sends ω_n to 1 and all other fundamental weights to zero. Consequently, it suffices to consider the term $2\omega_{n-1}\omega_n$ in q. But

$$\begin{aligned} 2c_1^{K_0}([\mathcal{L}(\omega_n)]) &\equiv 2\big(c_1^{K_0}(g_{s_n}) + c_1^{K_0}([\mathcal{L}(\alpha_n)])\big) \mod \gamma^2(\mathfrak{B}_s) \\ &\equiv c_1^{K_0}(2g_{s_n}) + 2c_1^{K_0}([\mathcal{L}(\alpha_n)]), \end{aligned}$$

and again we find that $\mathrm{c}'(\theta)$ is rational.

Type E_6. If G has type E_6, Λ/Λ_r equals $\mathbb{Z}/3$. Then $p = 3$ and we have

$$q = \sum_{i=1}^{6} \omega_i^2 - \omega_1\omega_3 - \omega_3\omega_4 - \omega_2\omega_4 - \omega_4\omega_5 - \omega_5\omega_6.$$

Note that ω_2 and ω_4 belong to the root lattice, so we can discard the ω_2^2, ω_4^2, and $\omega_2\omega_4$ terms. Similarly, ω_4 and $\omega_3 + \omega_5$ are both in the root lattice, eliminating the $\omega_4\omega_3 + \omega_4\omega_5$ term. Hence, we are left with

$$\omega_1^2 + \omega_3^2 + \omega_5^2 + \omega_6^2 - \omega_1\omega_3 - \omega_5\omega_6.$$

But $\omega_1 + \omega_6$ and $\omega_3 + \omega_5$ are in the root lattice, hence the term

$$(\omega_1 + \omega_6)^2 + (\omega_3 + \omega_5)^2 = \omega_1^2 + \omega_3^2 + \omega_5^2 + \omega_6^2 + 2\omega_1\omega_6 + 2\omega_3\omega_5$$

corresponds to a rational cycle. Taking a difference with the previous line, we are reduced to checking if $2\omega_1\omega_6 + 2\omega_3\omega_5 + \omega_1\omega_3 + \omega_5\omega_6$ gives a rational cycle.

Now $x := \omega_1 - \omega_5$ and $y := \omega_3 - \omega_6$ are in the root lattice. Substituting ω_1 by $\omega_5 + x$ and ω_3 by $\omega_6 + y$ and expanding, we reduce to checking that $6\omega_5\omega_6 + 3x\omega_6 + 3y\omega_5 + xy$ corresponds to a rational cycle. But $3\omega_6$ and $3\omega_5$ correspond to rational cycles by Panin [25]. Therefore, to finish the proof we have to check that the cycle $6c_1^{K_0}(\mathcal{L}(\omega_5))c_1^{K_0}(\mathcal{L}(\omega_6))$ corresponding to $6\omega_5\omega_6$ is rational.

Since $\omega_5 + \omega_6$ is in the root lattice, the cycle $c_1^{K_0}([\mathcal{L}(\omega_5 + \omega_6)])$ is rational. Therefore, by the formula for the first Chern class

$$6c_1^{K_0}(\mathcal{L}(\omega_5) \otimes \mathcal{L}(\omega_6)) = 6c_1^{K_0}(\mathcal{L}(\omega_5)) + 6c_1^{K_0}(\mathcal{L}(\omega_6)) - 6c_1^{K_0}(\mathcal{L}(\omega_5))c_1^{K_0}(\mathcal{L}(\omega_6)).$$

But the left-hand side is rational as well as the terms $6c_1^{K_0}(\mathcal{L}(\omega_5))$, $6c_1^{K_0}(\mathcal{L}(\omega_6))$, hence, the desired cycle is rational. □

4.4. Essential dimension. We now apply Proposition 4.4 to strengthen a well-known application of the Rost invariant to give a lower bound on the essential dimension $\operatorname{ed}(G)$ for some algebraic groups G. (We refer to Reichstein's 2010 ICM lecture [29] for a definition and survey of essential dimension. Roughly speaking, it gives the number of parameters required to specify a G-torsor.)

The following result is well known.

Proposition 4.5. *Let G be an absolutely almost simple algebraic group that is simply connected. Then* $\operatorname{ed}(G) \geq 3$ *unless G is isomorphic to* Sp_{2n} *for some $n \geq 2$ or* SL_n *(in which case* $\operatorname{ed}(G) = 0$*).*

Proof. Because $N(G) \neq 1$, the Rost invariant is a nonconstant invariant with values in $H^3(*, \mathbb{Q}/\mathbb{Z}(2))$. □

This lower bound on essential dimension is weak, but the proof has the advantage of being uniform – it doesn't depend on the characteristic and depends rather little on the type of G – so this proposition and its proof are standard in the theory. The results of this section allow us to remove the hypothesis "simply connected".

Proposition 4.6. *Let G be an absolutely almost simple algebraic group which is not of type A or C. Then* $\operatorname{ed}(G) \geq 3$.

Proof. As essential dimension only goes down with field extensions, we may assume that k is algebraically closed and bound $\mathrm{ed}(G_s)$ where G_s is a split simple group not of type A or C. Put $\widetilde{G}_s$ for the simply connected cover of G_s. Fix a versal $\widetilde{G}_s$-torsor $\tilde{\xi} \in H^1(L, \widetilde{G}_s)$ for some extension L/k. Let K be a field between k and L of minimal transcendence degree such that there is a $\xi \in H^1(K, G_s)$ whose image in $H^1(L, G_s)$ is the same as the image of $\tilde{\xi}$.

For sake of contradiction, suppose that K has transcendence degree at most 2 over k. By the hypothesis on the type of G, the Tits algebras of ${}_\xi G$ have exponent a power of $p = 2$ or 3 and so are actually of index 1 or p over K by [1, p. 208]. By Proposition 4.4, there is a class $\psi \in \gamma^2(\mathfrak{B})$ whose image under restriction to L is $\mathrm{c}'(\theta)$. Now $\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}_K)$ is zero by Proposition 3.2 because $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is zero, and it follows that $\mathrm{c}'(\theta)$ is zero in $\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}_L)$. But $\mathrm{c}'(\theta)$ has order $N(G_s) \neq 1$, a contradiction. □

This result can also be proved on a case-by-case basis using known techniques. However, our proof shows how a typical application of the Rost invariant can be strengthened via our Proposition 4.6, which itself is merely an application of our core result, Theorem 3.1.

5. Examples of torsion in CH^2 and CH^3

In the present section we apply the results of Section 3 to describe the torsion of CH^2 and CH^3 of projective homogeneous varieties and the associated generalized Rost motives. We maintain the notation of Section 3.

Let X be a projective homogeneous G-variety such that G is split over $k(X)$. ("X is generically split.") Thanks to Proposition 3.2, we may view $\operatorname{Tors}\mathrm{CH}^2(X)$ as known, so we now investigate $\operatorname{Tors}\mathrm{CH}^3(X)$. Let n denote the rank of G and let r denote the rank of the Picard group of X over an algebraic closure of k. For an abelian group A and a prime p, write $\operatorname{Tors}_p A$ for the subgroup of A consisting of elements of order a power of p.

We remark that the results below only use the fact that $\operatorname{Tors}\mathrm{CH}^2(X)$ is cyclic of order dividing $N(G)$, which follows from our Theorem 3.1 (i) and sequence (1). They do not need the finer result of Proposition 3.2, hence also do not need material from [13] and [27].

We will use the following well-known fact.

Lemma 5.1. *The restriction of the m-th Chern class gives a surjection*

$$\operatorname{Tors}\tau^{m/m+1}(X) \twoheadrightarrow (m-1)!\operatorname{Tors}\mathrm{CH}^m(X)$$

and for each prime p not dividing $(m-1)!$, c_m is an isomorphism

$$\operatorname{Tors}_p \tau^{m/m+1}(X) \xrightarrow{\simeq} \operatorname{Tors}_p \mathrm{CH}^m(X).$$

Proof. By Riemann–Roch (see Section 1.1), the composition

$$\mathrm{CH}^m(X) \overset{\mathrm{pr}}{\twoheadrightarrow} \tau^{m/m+1}(X) \xrightarrow{c_m} \mathrm{CH}^m(X)$$

is multiplication by $(-1)^{m-1}(m-1)!$, hence $c_m(\tau^{m/m+1}(X))$ is $(m-1)!\,\mathrm{CH}^m(X)$. For $x \in \operatorname{Tors}\mathrm{CH}^m(X)$, we have $(m-1)! \cdot x = c_m(\mathrm{pr}(x))$, where $\mathrm{pr}(x)$ is in $\operatorname{Tors}\tau^{m/m+1}(X)$. This proves the first claim, from which the second claim follows immediately. □

Proposition 5.2. *The subgroup*

$$4 \cdot \frac{N(G)}{o(r(G))} \cdot \operatorname{Tors} \mathrm{CH}^3(\mathfrak{B})$$

is a quotient of $(\mathbb{Z}/N(G)\mathbb{Z})^{\oplus n}$. *In particular, if* $o(r(G)) = N(G)$, *then the torsion part of* $\mathrm{CH}^3(\mathfrak{B})$ *can consist only of subgroups* $\mathbb{Z}/2^s\mathbb{Z}$ *for* $s \leq 4$, $\mathbb{Z}/3\mathbb{Z}$, *or* $\mathbb{Z}/5\mathbb{Z}$.

Proof. Put $q := N(G)/o(r(G))$. On the one hand, the subgroup $2 \operatorname{Tors} \gamma^{3/4}(\mathfrak{B})$ maps into $2 \operatorname{Tors} \tau^{3/4}(\mathfrak{B})$, and the image contains $2q \operatorname{Tors} \tau^{3/4}(\mathfrak{B})$ by Theorem 3.1 (ii). Part (iii) of that theorem gives that $2q \operatorname{Tors} \tau^{3/4}(\mathfrak{B})$ is a quotient of $(\mathbb{Z}/N(G))^{\oplus n}$.

On the other hand, $2q \operatorname{Tors} \tau^{3/4}(\mathfrak{B})$ surjects onto $4q \operatorname{Tors} \mathrm{CH}^3(\mathfrak{B})$ by Lemma 5.1; this proves the first claim. The second claim follows immediately. □

We can also control the torsion on $\mathrm{CH}^3(X)$ based on information about the torsion in $\mathrm{CH}^2(X)$ and the motivic decomposition of X, as we now illustrate. Fix a prime p. In the category of Chow motives with $\mathbb{Z}/p\mathbb{Z}$-coefficients, the motive of X is a direct sum of shifts of an indecomposable motive $\mathcal{R}$, see [26, Theorem 5.17], where $\mathcal{R}$ depends on G but not the choice of X, see [26, Theorem 3.7]. We write $\mathrm{Ch}^m(\mathcal{R})$ for the m-th Chow group of $\mathcal{R}$ with $\mathbb{Z}/p\mathbb{Z}$ coefficients.

Lemma 5.3. *We have:*

(i) $(\operatorname{Tors}_p \mathrm{CH}^2(X)) \otimes \mathbb{Z}/p\mathbb{Z} \simeq \mathrm{Ch}^2(\mathcal{R})$;

(ii) $(\operatorname{Tors}_p \mathrm{CH}^3(X)) \otimes \mathbb{Z}/p\mathbb{Z} \simeq (\mathrm{Ch}^2(\mathcal{R}))^{\oplus r} \oplus \mathrm{Ch}^3(\mathcal{R})$.

(iii) $\operatorname{Tors} \mathrm{CH}^3(\mathfrak{B}) \simeq (\operatorname{Tors} \mathrm{CH}^2(X))^{\oplus(n-r)} \oplus \operatorname{Tors} \mathrm{CH}^3(X)$.

We remark that some results of Merkurjev–Suslin [24] and Yagita [33, Theorem 10.5, Corollary 10.8] include a calculation of $\mathrm{Ch}^2(\mathcal{R})$. Part (i) in the lemma (combined with Theorem 3.1 (i)) also does this and furthermore produces a generator, namely the special cycle θ.

Proof of Lemma 5.3. The expression of the motive of X from [26] gives

$$(\operatorname{Tors}_p \mathrm{CH}^m(X)) \otimes \mathbb{Z}/p\mathbb{Z} \simeq \overline{\mathrm{Ch}}^m(\mathcal{R}) \oplus (\overline{\mathrm{Ch}}^{m-1}(R))^{\oplus r} \oplus (\overline{\mathrm{Ch}}^{m-2}(\mathcal{R}))^{\oplus \cdots} \oplus \cdots,$$

where $\overline{\mathrm{Ch}}^m(\mathcal{R})$ denotes the kernel of the restriction $\mathrm{Ch}^m(\mathcal{R}) \to \mathrm{Ch}^m(\mathcal{R} \times_k \bar{k})$ to the algebraic closure $\bar{k}$. By the formula for the generating function [26, Theorem 5.13 (3)] and Table 4.13 in ibid., we have $\overline{\mathrm{Ch}}^0(\mathcal{R}) = \overline{\mathrm{Ch}}^1(\mathcal{R}) = 0$ and $\overline{\mathrm{Ch}}^i(\mathcal{R}) = \mathrm{Ch}^i(\mathcal{R})$ for $i = 2, 3$. This implies claims (i) and (ii).

Claim (iii) is proved similarly, but using the integral motivic decomposition from [26, Theorem 3.7] with $Y = \mathfrak{B}$. □

Proposition 5.4. *Fix an odd prime* p. *If* $\operatorname{Tors}_p \mathrm{CH}^2(X) \neq 0$, *then*

(i) $p = 3$ *or* 5;

(ii) $\mathrm{Ch}^2(\mathcal{R}) \simeq \mathbb{Z}/p\mathbb{Z}$ *and* $\mathrm{Ch}^3(\mathcal{R}) = 0$;

(iii) $\operatorname{Tors}_p \mathrm{CH}^2(X) \simeq \mathbb{Z}/p\mathbb{Z}$ *and* $\operatorname{Tors}_p \mathrm{CH}^3(X) \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus r}$.

Proof. By Proposition 3.2 (or [26, Theorem 3.7]), $\mathrm{CH}^2(X)$ and $\mathrm{CH}^2(\mathfrak{B})$ have the same p-torsion. As $\mathrm{Tors}\,\mathrm{CH}^2(\mathfrak{B})$ has order dividing $N(G)$ by Theorem 3.1 (i), the list of Dynkin indexes in Section 1.4 gives that $p = 3$ or 5 and $\mathrm{Tors}_p\,\mathrm{CH}^2(\mathfrak{B}) \simeq \mathbb{Z}/p\mathbb{Z}$. Combining this with Lemma 5.3 (i), it only remains to prove the claims about $\mathrm{Ch}^3(\mathcal{R})$ and $\mathrm{CH}^3(X)$.

Tensoring sequence (1) with $\mathbb{Z}/p\mathbb{Z}$, we find that

$$\gamma^3(\mathfrak{B}) \otimes \mathbb{Z}/p\mathbb{Z} = \tau^3(\mathfrak{B}) \otimes \mathbb{Z}/p\mathbb{Z}.$$

Combining Lemma 5.1 and Theorem 3.1 (iii) gives that $\mathrm{Tors}_p\,\mathrm{CH}^3(\mathfrak{B})$ is a product of at most n copies of $\mathbb{Z}/p\mathbb{Z}$. By Lemma 5.3 (ii) applied to $X = \mathfrak{B}$ we obtain

$$(\mathrm{Tors}_p\,\mathrm{CH}^3(\mathfrak{B})) \otimes \mathbb{Z}/p\mathbb{Z} \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus n} \oplus \mathrm{Ch}^3(\mathcal{R}).$$

Since the right-hand side already contains n copies of $\mathbb{Z}/p\mathbb{Z}$, $\mathrm{Tors}_p\,\mathrm{CH}^3(\mathfrak{B}) = (\mathbb{Z}/p\mathbb{Z})^{\oplus n}$ and $\mathrm{Ch}^3(\mathcal{R})$ is zero. The second part of (iii) now follows by Lemma 5.3 (iii). □

This proposition gives the claim in the example from the introduction.

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Skip Garibaldi, Department of Mathematics & Computer Science, Emory University,
Atlanta, GA 30322, USA
e-mail: skip@member.ams.org

Kirill Zainoulline, Department of Mathematics and Statistics, University of Ottawa,
585 King Edward, Ottawa ON K1N6N5, Canada
e-mail: kirill@uottawa.ca

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