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Essential dimension of algebraic groups, including bad characteristic

Skip Garibaldi and Robert M. Guralnick

Abstract. We give upper bounds on the essential dimension of (quasi-) simple algebraic groups over an algebraically closed field that hold in all characteristics. The results depend on showing that certain representations are generically free. In particular, aside from the cases of spin and half-spin groups, we prove that the essential dimension of a simple algebraic group G of rank at least two is at most $\dim G - 2(\operatorname{rank} G) - 1$. It is known that the essential dimension of spin and half-spin groups grows exponentially in the rank. In most cases, our bounds are as good as or better than those known in characteristic zero and the proofs are shorter. We also compute the generic stabilizer of an adjoint group on its Lie algebra.

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1. Introduction. The essential dimension of an algebraic group G is the minimal transcendence degree of the field of definition of a versal G-torsor. (Although inaccurate, one can think of it as the number of parameters needed to specify a G-torsor.) This invariant captures deep information about algebraic structures with automorphism group G, and it is difficult to calculate. For example, the fact that $\operatorname{ed}(\mathrm{PGL}_2) = \operatorname{ed}(\mathrm{PGL}_3) = 2$ corresponds to the classical fact that division algebras of dimension 2^2 or 3^2 over their center are cyclic, and it is an open problem whether the essential dimension of PGL_p is 2 for primes $p \geq 5$ [1, Problem 6.2], although it is known that $\operatorname{ed}(\mathrm{PGL}_n)$ is not $O(n)$

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[31]. Therefore, the bulk of known results on essential dimension provides upper or lower bounds, as in [4,12,18,26], etc. (see [32,33], or [38] for a survey of the current state of the art.) In this paper, we provide upper bounds on $\mathrm{ed}(G)$ for every simple algebraic group G over an algebraically closed field k, regardless of the characteristic of k. Our bounds are in some cases as good as (Theorem 1.4) or better (Theorems 1.1 and 1.3) than the bounds known in characteristic zero, and have shorter proofs. One summary consequence of our results is the following.

Theorem 1.1. *Let G be a simple algebraic group over an algebraically closed field. Then*

$$\mathrm{ed}(G) \le \dim G - 2(\operatorname{rank} G) - 1,$$

$G \cong \mathrm{PGL}_2$ *or* $G \cong \mathrm{Spin}_n$ *or* HSpin_n *for some* n*.*

For the excluded cases, $\mathrm{ed}(\mathrm{PGL}_2) = 2$ (Example 6.3). For spin and half-spin groups, essential dimension grows exponentially in n [9] whereas the dimension, $\binom{n}{2}$, is quadratic in n. Specifically, $\mathrm{ed}(\mathrm{Spin}_n) > \dim \mathrm{Spin}_n$ for all $n \ge 19$ and $\mathrm{ed}(\mathrm{HSpin}_n) > \dim \mathrm{HSpin}_n$ for n divisible by 4 and ≥ 20.

Adjoint groups. Under the additional hypotheses that G is adjoint and $\operatorname{char} k = 0$, it is well known that an adjoint semisimple group G acts generically freely on $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$[1] and consequently $\mathrm{ed}(G) \le \dim G$, as was pointed out in [9, Remark 3-11]. In this setting, the stronger bound in Theorem 1.1 was proved in [24]. Dropping the hypothesis on $\operatorname{char} k$ but still assuming G is adjoint, the bound $\mathrm{ed}(G) \le \dim G - 2(\operatorname{rank} G)$ was recently proved in [7, Cor. 10].

Theorem 1.1 for adjoint groups includes the following bounds, where we write T_n^{adj} for an adjoint group of type T_n:

$$\begin{aligned} &\mathrm{ed}(E_6^{\mathrm{adj}}) \le 65, \quad \mathrm{ed}(E_7^{\mathrm{adj}}) \le 118, \quad \mathrm{ed}(E_8) \le 231, \\ &\text{and} \quad \mathrm{ed}(D_n^{\mathrm{adj}}) \le 2n^2 - 3n - 1 \quad \text{for } n \ge 4. \end{aligned} \tag{1.2}$$

(The adjoint group D_n^{adj} is sometimes denoted by PSO_{2n}.) These bounds agree with those in [24] for characteristic 0. (The number 112 given there for E_7^{adj} was a typo.)

We remark that the essential dimension of SO_{2n+1} (adjoint of type B_n) is $2n$ if $\operatorname{char} k \ne 2$ [37] and is $n+1$ if $\operatorname{char} k = 2$ [5]. For SO_{2n} (of type D_n), the essential dimension is $2n-1$ if $\operatorname{char} k \ne 2$ and is n or $n+1$ if $\operatorname{char} k = 2$.

Groups of type C. We give the following upper bound for adjoint groups of type C_n, which improves on the bound $2n^2-3n-1$ given in [24] in characteristic zero.

Theorem 1.3. *Over an algebraically closed field k and for $n \ge 4$*

$$\mathrm{ed}(\mathrm{GL}_{2n}/\mu_2) \le \mathrm{ed}(\mathrm{PSp}_{2n}) \le \begin{cases} 2n^2 - 3n - 4 & \textit{if } \operatorname{char} k \nmid n \textit{ or } n = 4 \\ 2n^2 - 3n - 6 & \textit{if } \operatorname{char} k \mid n \textit{ and } n > 4. \end{cases}$$

[1] By, e.g., [39, Lemma 3.3(b)]. For analogous statements in prime characteristic, see Section 9.

The interesting case of Theorem 1.3 is when n is even; in the special case where n is odd, the natural map $\mathrm{PGL}_2 \times \mathrm{SO}_n \hookrightarrow \mathrm{PSp}_{2n}$ gives a surjection $H^1(k, \mathrm{PGL}_2) \times H^1(k, \mathrm{SO}_n) \to H^1(k, \mathrm{PSp}_{2n})$, and $\mathrm{ed}(\mathrm{PSp}_{2n}) = n + 1$ for n odd, cf. [28, p. 302].

It is known that $\mathrm{ed}(\mathrm{GL}_8 / \mu_2) = 8$ if $\mathrm{char}\, k \neq 2$ [4, Cor. 1.4] and is ≤ 10 if $\mathrm{char}\, k = 2$ [3, Cor. 1.4], somewhat better than the bound ≤ 16 provided by Theorem 1.3.

Groups of type A. The essential p-dimension of GL_n / μ_m and of SL_n / μ_m have been studied in [3,4,11]. Here and in (6.7) we give upper bounds for the essential dimension (without the p).

Theorem 1.4. *Over an algebraically closed field and for m dividing $n \geq 4$, we have*

$$\mathrm{ed}(\mathrm{PGL}_n) \leq n^2 - 3n + 1 \quad \textit{and} \quad \mathrm{ed}(\mathrm{SL}_n / \mu_m) \leq n^2 - 3n + n/m + 1.$$

If $m = 1$, then $\mathrm{ed}(\mathrm{SL}_n) = 0$. If $m = n$, then $\mathrm{SL}_n / \mu_m = \mathrm{PGL}_n$. Our bound for PGL_n agrees with the one given by [24] in $\mathrm{char}\, k = 0$; we remove this hypothesis. A better bound on $\mathrm{ed}(\mathrm{PGL}_n)$ is known for n odd [27]. See [32, §10] or [38, §7.6] for discussions of many more results on upper bounds for PGL_n. If $m = 2$, then (applying Lemma 6.4) the bound in Theorem 1.3 is better by about a factor of 2.

Exceptional groups. Concerning exceptional groups, a series of papers [24,25, 29,30] have led to the following upper bounds for exceptional groups:

$$\mathrm{ed}(F_4) \leq 7, \quad \mathrm{ed}(E_6^{\mathrm{sc}}) \leq 8, \quad \text{and} \quad \mathrm{ed}(E_7^{\mathrm{sc}}) \leq 11 \quad \text{if } \mathrm{char}\, k \neq 2, 3.$$

(Here F_4, E_6^{sc}, and E_7^{sc} denote simple and simply connected groups of types F_4, E_6, and E_7; the displayed upper bounds are meant to be compared with the dimensions of 52, 78, and 133, respectively. These upper bounds are close to the known lower bounds of 5, 4, and 8 for $\mathrm{char}\, k \neq 2, 3$.) The proofs of these upper bounds for F_4 and E_7^{sc} are technical and detailed calculations. The following weaker bounds have the advantage of simple proofs and holding for fields of characteristic 2 and 3.

Theorem 1.5. *Over an algebraically closed field, we have*

$$\mathrm{ed}(F_4) \leq 19, \quad \mathrm{ed}(E_6^{\mathrm{sc}}) \leq 20, \quad \textit{and} \quad \mathrm{ed}(E_7^{\mathrm{sc}}) \leq 49.$$

The proofs of most of the theorems above rely on computations of the (scheme-theoretic) stabilizer of a generic element in a representation of $N_G(T)$ for T a maximal torus in G. The proof of Theorem 1.3 uses the computation of a generic stabilizer in the action of Sp on $L(\lambda_2)$. Using the same technique, we calculate the generic stabilizer of an adjoint group G acting on its Lie algebra. In particular, this stabilizer is connected unless $\mathrm{char}\, k = 2$. In the final section, we give upper bounds on essential dimension for small spin and half-spin groups, completing the list of upper bounds on $\mathrm{ed}(G)$ for G simple over an algebraically closed field. (An algebraic group over an algebraically closed field is *simple*—sometimes called quasi-simple—if it has no proper connected normal subgroups and is perfect. In particular, a simple algebraic group can have a finite center.)

2. Generically free actions. Let G be an affine group scheme of finite type over a field k, which we assume as algebraically closed. (If G is additionally smooth, then we say that G is an *algebraic group*.) We put G° for the identity component of G. If G acts on a variety X, the stabilizer G_x of an element $x \in X(k)$ is a sub-group-scheme of G with points

$$G_x(R) = \{g \in G(R) \mid gx = x\}$$

for every k-algebra R. The Statement "for generic x" means that there is a dense open subset U of X such that the property holds for all $x \in U$.

Suppose G acts on a variety X in the sense that there is a map of k-schemes $G \times X \to X$ satisfying the axioms of a group action. We say that G acts *generically freely* on X if there is a nonempty open subset U of X such that for every $u \in U$ the stabilizer G_u is the trivial group scheme 1. It is equivalent to require that $G_u(k) = 1$ and $\mathrm{Lie}(G_u) = 0$. Indeed, if $\mathrm{Lie}(G_u) = 0$, then G_u is finite étale and (since k is algebraically closed) $G_u(k) = 1$ implies that $G_u = 1$.

Example 2.1. For T a diagonalizable group scheme (e.g., a split torus) acting linearly on a vector space V, the stabilizer T_v of a generic vector $v \in V$ is $\cap_{\omega \in \Omega} \ker \omega$ where $\Omega \subset T^*$ is the set of weights of V, i.e., T_v is the kernel of the action. (By the duality between diagonalizable group schemes and finitely generated abelian groups, this is a statement about group schemes.) In particular, T acts generically freely on V if and only if Ω spans T^*.

Similarly, the stabilizer $T_{[v]}$ of a generic element $[v] \in \mathbb{P}(V)$ is $\cap_{\omega,\omega'} \ker(\omega - \omega')$, so T acts generically freely on $\mathbb{P}(V)$ iff the set of differences $\omega - \omega'$ span T^*.

For other groups G, we have the following well-known lemma, see for example [16, Lemma 2.2].

Lemma 2.2. *Suppose G is connected and X is irreducible. If there is a field $K \supseteq k$ and an element $x_0 \in X(K)$ such that G_{x_0} is finite étale, then there is an $n \geq 1$ and a nonempty open $U \subseteq X$ such that, for every algebraically closed field $E \supseteq k$ and every $u \in U(E)$, G_u is finite étale and $|G_u(E)| = n$.* □

Note that finding some x_0 with $G_{x_0} = 1$ does not imply that G acts generically freely on X; it is common that such an x_0 will exist in cases where G_x is finite étale but $\neq 1$ for generic x. This was pointed out already in [2]; see [21] for more discussion and examples.

Nonetheless, Lemma 2.2 may be used to prove that an action is generically free as follows. Suppose G, X, and the action of G on X can be defined over a countable algebraically closed field F and that X is unirational, i.e., there is an F-defined dominant rational map $\phi : \mathbb{A}^d \dashrightarrow X$ for some d. Adjoin d indeterminates $a_1, \ldots, a_d$ to F and calculate G_{x_0} for $x_0 = \phi(a_1, \ldots, a_d)$. As F is countable, for K an uncountable algebraically closed field containing F, the elements of $\mathbb{A}^d(K)$ with algebraically independent coordinates are the complement of countably many closed subsets, so they are dense. Therefore, modifying ϕ by an F-automorphism of $\mathbb{A}^d$, the calculation of G_{x_0} implicitly also calculates G_x for x in a dense subset. In particular, if $G_{x_0} = 1$, then the lemma gives that G acts generically freely on X.

Groups whose identity component is a torus. Suppose that G is an algebraic group whose identity component is a torus T. As k is assumed algebraically closed, the component group G/T is a finite constant group. We are interested in representations V of G such that G acts generically freely on V or $\mathbb{P}(V)$. Evidently, it is necessary that T acts faithfully on V or $\mathbb{P}(V)$, respectively.

Lemma 2.3. *Let G be an algebraic group with identity component a torus T. Suppose that G acts linearly on a vector space V such that*

1. *every weight of V has multiplicity* 1, *and*
2. *for Ω the set of weights of V, G/T acts faithfully on the kernel of the map $\psi\colon \oplus_{\omega\in\Omega}\mathbb{Z} \mapsto T^*$ given by $(n_\omega) \mapsto \sum_\omega n_\omega \omega$.*

If T acts faithfully on V (resp., $\mathbb{P}(V)$), then G acts generically freely on V (resp., $\mathbb{P}(V)$).

We give a concrete proof. Alternatively, one could adapt the proof of [34, Lemma 3.3].

Proof. As G is the extension of a finite constant group by a torus, it and the representation V are defined over the algebraic closure of the prime field in k. Put K for the algebraic closure of the field obtained by adjoining independent indeterminates c_χ to k for each weight χ of V. Fix elements $v_\chi \in V$ generating the χ weight space for each χ and put $v := \sum_\chi c_\chi v_\chi \in V \otimes K$.

If T acts faithfully on $\mathbb{P}(V)$, put $S := G_{[v]}$. Otherwise T acts faithfully on V, and we put $S := G_v$. By the discussion following Lemma 2.2, it suffices to show that $S = 1$. Now, $\mathrm{Lie}(S)$ is contained in the subalgebra of $\mathrm{Lie}(G) = \mathrm{Lie}(T)$ stabilizing $[v]$, resp. v, which is zero by hypothesis, so S is finite étale. As $S \cap T = 1$ by Example 2.1, the map $G \to G/T$ restricts to an isomorphism from S to its image.

For sake of contradiction, suppose there exists an $s \in S(K)$ mapping to a non-identity element w in $(G/T)(K)$. Pick $n \in G(k)$ with the same image w, so $s = nt$ for some $t \in T(K)$. Now $nv_\chi = m_\chi v_{w\chi}$ for some $m_\chi \in k^\times$, and we have an equation

$$sv = ntv = \sum_\chi c_\chi \chi(t) m_\chi v_{w\chi},$$

hence $\chi(t) = \delta(s) c_{w\chi}/(c_\chi m_\chi)$ for all χ, where $\delta(s) := sv/v \in K^\times$. As S is finite étale, the homomorphism $\delta\colon S \to \mathbb{G}_m$ has image the a-th roots of unity for some a, hence $\delta(s) \in k^\times$.

By hypothesis, there exist $\chi_1, \ldots, \chi_r \in \Omega$ and nonzero $z_1, \ldots, z_r \in \mathbb{Z}$ such that $\sum z_i \chi_i = 0$ in T^*, yet the tuple $(z_\chi) \in \oplus_{\chi\in\Omega}\mathbb{Z}$ is not fixed by w where

$z_\chi = z_i$ if $\chi = \chi_i$ and $z_\chi = 0$ otherwise. As $\sum z_i\chi_i = 0$, we have

$$1 = \prod_i \left(\frac{\delta(s) c_{w\chi_i}}{c_{\chi_i} m_{\chi_i}} \right)^{z_i}, \tag{2.4}$$

an equation in K, where $\delta(s)$ and the m_{χ_i} belong to $k^\times$. But the indeterminates appearing in the numerator correspond to the tuple $(z_{w\cdot\chi})$ whereas those in the denominator correspond to (z_χ), so the equality (2.4) is impossible and $S(K) = 1$. □

Example 2.5. Suppose there is an element $-1 \in (G/T)(k)$ that acts by -1 on T^*. Then we may partition $\Omega \setminus \{0\}$ as $P \coprod -P$ for some set P. *If* $|P| > \dim T$, *then* -1 *acts nontrivially on* $\ker\psi$. Indeed, there are $n_\pi \in \mathbb{Z}$ for $\pi \in P$, not all zero, so that $\sum n_\pi \pi = 0$ in T^*, which provides an element of $\ker\psi$ that is moved by -1.

The group $\mathbf{AGL_1}$. The following result will be used for groups of type C.

Let k be an algebraically closed field of characteristic $p \geq 0$. Let X be the variety of monic polynomials of degree n over k. Of course, X is isomorphic to affine space $\mathbb{A}^n$ and can also be identified with $\mathbb{A}^n/S_n$ (where the coordinates are just the roots of the polynomial). Let X_0 be the subvariety of X such that the coefficient of x^{n-1} is 0 (i.e., the sum of the roots of f is 0). Let $G = \mathrm{AGL}_1$, the group with k-points $\{\left(\begin{smallmatrix} c & b \\ 0 & 1 \end{smallmatrix}\right) \mid c \in k^\times, b \in k\}$, so G is a semidirect product $\mathbb{G}_m \ltimes \mathbb{G}_a$ and is isomorphic to a Borel subgroup of PGL_2. An element $g \in G$ acts on $\mathbb{A}^1$ by $y \mapsto cy + b$ and we can extend this to an action on X (by acting on each root of f). Note that G preserves X_0 if and only if p divides n. In any case $\mathbb{G}_m$ does act on X_0.

Lemma 2.6. *If p does not divide $n > 2$, then $\mathbb{G}_m$ acts generically freely on X_0. If p divides n and $n > 4$, then G acts generically freely on X_0.*

Proof. We just give the proof of the group of k-points. The proof for the Lie algebra is identical.

Note that if $c = (0, c) \in k^\times$ and f has distinct roots, then c fixes f implies that $c^{n(n-1)} = 1$ since c preserves the discriminant of f. In particular, there are only finitely many possibilities for c.

Note that the dimension of the fixed point space of multiplication by c on X has dimension at most $n/2$ and so the fixed point space on X_0 is a proper subvariety (because $n > 2$) and has codimension at least 2 if $n > 4$.

If p does not divide n, then we see that there are only finitely many elements of $k^\times$ which have a fixed space which intersects the open subvariety of X_0 consisting of elements with nonzero discriminant. Thus, the finite union of these fixed spaces is contained in a proper subvariety of X_0 whence for a generic point the stabilizer is trivial.

Now suppose that p divides n. Then translation by b has a fixed space of dimension $n/p \leq n/2$ on X and so similarly the fixed space on X_0 has codimension at least 2 for $n > 4$.

There is precisely one conjugacy class of nontrivial unipotent elements in G and this class has dimension 1. Thus the union of all fixed spaces of nontrivial unipotent elements of G is contained in a hypersurface for $n > 4$. Any semisimple element of G is conjugate to an element of $k^\times$ (i.e., to an element of the form $(0, c)$), and so there are only finitely many such conjugacy classes which have fixed points on the locus of polynomials with nonzero discriminant. Again, since each class is 1-dimensional and each fixed space has codimension greater than 1, we see that the union of all fixed spaces is contained in a hypersurface of X_0 for $n > 4$. $\square$

If $n = 4$ and $p = 2$, then any $f \in X_0$ is fixed by a translation and so the action is not generically free.

3. Essential dimension. The essential dimension of an affine group scheme G over a field k can be defined as follows. For each extension K of k, write $H^1(K, G)$ for the cohomology set relative to the fppf (= faithfully flat and finitely presented) site as defined in, for example, [45]. For each $x \in H^1(K, G)$, we define $\mathrm{ed}(x)$ to be the minimum transcendence degree of K_0 over k for $k \supseteq K_0 \supseteq K$ such that x is in the image of $H^1(K_0, G) \to H^1(K, G)$. The *essential dimension* of G, denoted $\mathrm{ed}(G)$, is defined to be $\max \mathrm{ed}(x)$ as x varies over all extensions K of k and all $x \in H^1(K, G)$.

If V is a representation of G on which G acts generically freely, then $\mathrm{ed}(G) \leq \dim V - \dim G$, see, e.g., [32, Prop. 3.13]. We can decrease this bound somewhat by employing the following.

Lemma 3.1. *Suppose V is a representation of an algebraic group G. If there is a G-equivariant dominant rational map $V \dashrightarrow X$ for a G-variety X on which G acts generically freely, then* $\mathrm{ed}(G) \leq \dim X - \dim G$.

Proof. Certainly, G must act generically freely on V. In the language of [14] or [32, p. 424], then V is a versal and generically free G-variety and the natural map $V \dashrightarrow X$ is a G-compression. Therefore, referring to [32, Prop. 3.11], we find that $\mathrm{ed}(G) \leq \dim X - \dim G$. $\square$

4. The short root representation. Let G be an adjoint simple algebraic group and put V for the Weyl module with highest weight the highest short root. Fixing a maximal torus T in G, the weights of this representation are 0 (with some multiplicity) and the short roots Ω (each with multiplicity 1), and we put $\overline{V}$ for V modulo the zero weight space. It is a module for $N_G(T)$.

Proposition 4.1. *Suppose k is algebraically closed. If G is of type A_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, or F_4, then $N_G(T)$ acts generically freely on $\mathbb{P}(\overline{V})$ and*

$$\mathrm{ed}(N_G(T)) \leq |\Omega| - \dim T - 1.$$

The inequality in the proposition is reminiscent of the one in [24, Th. 1.3].

Example 4.2. The group PGL_n is adjoint of type A_{n-1}, and we identify it with the quotient of GL_n by the invertible scalar matrices. We may choose $T \subset \mathrm{PGL}_n$ to be the image of the diagonal matrices, and $N_G(T)$ is the image of the monomial matrices. The representation $\overline{V}$ is the space of matrices, with zeros on the diagonal, on which $N_G(T)$ acts by conjugation.

With this notation, for type A_1, the stabilizer in $N_G(T)$ of a generic element $v := \left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right)$ of $\overline{V}$ is $\mathbb{Z}/2$, with nontrivial element the image of v itself. So type A_1 is a genuine exclusion from the proposition.

Proof of Proposition 4.1. It suffices to prove that $N_G(T)$ acts generically freely, for then the inequality follows by Lemma 3.1. We apply Lemma 2.3.

For every short root α, there is a short root β such that $\langle \beta, \alpha \rangle = \pm 1$. (If α is simple, take β to be simple and adjacent to α in the Dynkin diagram. Otherwise, α is in the Weyl orbit of a simple root.) Thus, the kernel of $T \to \mathrm{PGL}(\overline{V})$ is contained in the kernel of $\beta - s_\alpha(\beta) = \pm\alpha$. As the lattice generated by the short roots α is the root lattice T^*, it follows that T acts generically freely on $\mathbb{P}(\overline{V})$.

So it suffices to verify Lemma 2.3(2). Fix $w \neq 1$ in the Weyl group; we find short roots $\chi_1, \ldots, \chi_r$ such that $\sum \chi_i = 0$ and the set $\{\chi_i\}$ is not w-invariant.

If $w = -1$, then take χ_1, χ_2 to be non-orthogonal short simple roots. They generate an A_2 subsystem and we set $\chi_3 := -\chi_1 - \chi_2$. (Alternatively, apply Example 2.5.) This proves the claim for type F_4: the kernel of the G/T-action on $\ker\psi$ is a normal subgroup of the Weyl group not containing -1, and therefore it is trivial.

If G has type A, D, or E, then all roots are short. As $w \neq \pm 1$, there is a short simple root χ_1 such that $w(\chi_1) \neq \pm\chi_1$. (Indeed, otherwise there would be simple roots α, α' such that $w(\alpha) = \alpha$, $w(\alpha') = -\alpha'$, yet α and α' are adjacent in the Dynkin diagram.) Take $\chi_2 = -\chi_1$.

For type C_n ($n \geq 3$), as in [6] we may view the root lattice $\mathbb{Z}[\Phi]$ as contained in a copy of $\mathbb{Z}^n$ with basis $\varepsilon_1, \ldots, \varepsilon_n$. If the kernel of the G/T-action on $\ker\psi$ does not contain -1, it contains the group H isomorphic to $(\mathbb{Z}/2)^{n-1}$ consisting of those elements that send $\varepsilon_i \mapsto -\varepsilon_i$ for an even number of indexes i and fix the others. Taking $\chi_1 = \varepsilon_1 - \varepsilon_2$, $\chi_2 = \varepsilon_2 - \varepsilon_3$, and $\chi_3 = -\chi_1 - \chi_2$ gives a set $\{\chi_i\}$ not stabilized by H. □

5. Groups of type C: proof of Theorem 1.3. Let G be the adjoint group of type C_n for $n > 3$ over an algebraically closed field k of characteristic p. Let $W := L(\omega_2)$ be the irreducible module for G with highest weight ω_2 where ω_2 is the second fundamental dominant weight (as numbered in [6]). We view W as the unique irreducible nontrivial G-composition factor of $Y := \wedge^2(V)$ where V is the natural module for Sp_{2n}. We recall that $Y = W \oplus k$ if p does not divide n. If p divides n, then Y is uniserial of length 3 with 1-dimensional socle and radical. Any element in Y has characteristic polynomial f^2 where f has degree n, and the radical Y_0 of Y is the set of elements with the roots of f summing to 0. (Note that aside from characteristic 2, Y_0 are the elements of trace 0 in Y.)

In particular, $\dim W = 2n^2 - n - 1$ if p does not divide n, and $\dim W = 2n^2 - n - 2$ if p does divide n.

As in [20], we view Y as the set of skew adjoint operators on V with respect to the alternating form defining Sp_{2n} with G acting as conjugation on Y.

Proposition 5.1. *If $n > 3$ and $(n,p) \neq (4,2)$, then G acts generically freely on $W \oplus W$ and on $\mathbb{P}(W) \times \mathbb{P}(W)$.*

Proof. Any element $y \in Y$ is conjugate to an element of the form $\operatorname{diag}(A, A^\top)$ acting on a direct sum of totally singular subspaces. A generic element of Y is thus an element where A is semisimple regular. Writing $V = V_1 \perp \cdots \perp V_n$ where the V_i are 2-dimensional nonsingular spaces on which y acts as a scalar, we see that a generic point of Y has stabilizer (as a group scheme) $\mathrm{Sp}_2^{\times n} = \mathrm{SL}_2^{\times n}$ in Sp_{2n} (and by [20], this is precisely the intersection of two generic conjugates of Sp_{2n} in SL_{2n}). The same argument shows that this is true for a generic point of Y_0.

In particular, if p does not divide n, the same is true for $W = Y_0$. It follows by Lemma 2.6 that for generic $w \in W$, $gw = cw$ for $g \in G$ and $c \in k^\times$ implies that $c = 1$. Thus, the stabilizer of a generic point in $\mathbb{P}(W)$ still has stabilizer $\mathrm{Sp}_2^{\times n} = \mathrm{SL}_2^{\times n}$.

If p does divide n, then $W = Y_0/k$ where we identify k with the scalar matrices in Y. We claim that (for $n > 4$) the generic stabilizer is still $\mathrm{Sp}_2^{\times n} = \mathrm{SL}_2^{\times n}$ on $\mathbb{P}(W)$. Again, this follows by Lemma 2.6 since if $gw = cw + b$ with $b, c \in k$ and $g \in G$, then for w generic, $b = 0$ and $c = 1$.

It is straightforward to see that the same is true for W, because for a generic point anything stabilizing y modulo scalars must stabilize y (see Lemma 2.6). Thus, in all cases, the generic stabilizer of a point in $\mathbb{P}(W) \times \mathbb{P}(W)$ is the same as for $Y \oplus Y$.

Consider GL_{2n} acting on $Y \oplus Y \oplus Y$. The stabilizer of a generic point of Y is clearly a conjugate of Sp_{2n}. It follows from [20] that the stabilizer of a generic element of $Y \oplus Y \oplus Y$ is central. (The result is only stated for the algebraic group but precisely the same proof holds for the group scheme.) Thus, the same holds for Sp_{2n} acting on $Y \oplus Y$ and so also on $\mathbb{P}(W) \times \mathbb{P}(W)$. The result follows. $\square$

We can now improve and extend Lemire's bound for $\operatorname{ed}(\mathrm{PSp}_{2n})$ from [24, Cor. 1.4] both numerically and to fields of all characteristics.

Proof of Theorem 1.3. For the first inequality, the group GL_{2n}/μ_2 has an open orbit on $\wedge^2(k^{2n})$, and the stabilizer of a generic element is PSp_{2n}. Consequently, the induced map $H^1(K, \mathrm{PSp}_{2n}) \to H^1(K, \mathrm{GL}_{2n}/\mu_2)$ is surjective for every field K, see [15, Th. 9.3] or [42, §III.2.1]. (Alternatively, the domain classifies pairs (A, σ) where A is a central simple algebra of degree $2n$ and exponent 2 and σ is a symplectic involution [23, 29.22], and the codomain classifies central simple algebras of degree $2n$ and exponent 2. The map is the forgetful one $(A, \sigma) \mapsto A$.) Thus $\operatorname{ed}(\mathrm{GL}_{2n}/\mu_2) \leq \operatorname{ed}(\mathrm{PSp}_{2n})$.

For the second inequality, assume that $n \geq 4$ and if $n = 4$, then $p \neq 2$. As PSp_n acts generically freely on $\mathbb{P}(W) \times \mathbb{P}(W)$, $\operatorname{ed}(G) \leq 2(\dim \mathbb{P}(W)) - \dim G$ by Lemma 3.1. Theorem 1.3 follows because $\dim \mathbb{P}(W) = 2n^2 - n - \delta$ where $\delta = 3$ if p divides n and 2 otherwise.

If $n = 4$ and $p = 2$, G still acts generically freely on $Y_0 \oplus Y_0$. Indeed, arguing as above we see that G acts generically freely on $\mathbb{P}(Y_0) \times \mathbb{P}(Y_0)$ and the result follows in this case. $\square$

6. Groups of type A: proof of Theorem 1.4. For the proofs of Theorems 1.1, 1.4, and 1.5, we use the fact that $\mathrm{ed}(N_G(T)) \geq \mathrm{ed}(G)$ because for every field $K \supseteq k$, the natural map $H^1(K, N_G(T)) \to H^1(K, G)$ is surjective (which in turn holds by [42, §III.2.2, Lemma 1] because for K separably closed, all maximal K-tori in G are $G(K)$-conjugate).

Let T be a maximal torus in $G := \mathrm{PGL}_n$ for some $n \geq 4$. The representation $\overline{V}$ of $N_G(T)$ from Section 4 may be identified with the space of n-by-n matrices with zeros on the diagonal. It decomposes as $\overline{V} = \oplus_{i=1}^n W_i$, where W_i is the subspace of matrices whose nonzero entries all lie in the i-th row; $N_G(T)$ permutes the W_i's.

Lemma 6.1. *If $n \geq 4$, then $N_G(T)$ acts generically freely on $X := \mathbb{P}(W_1) \times \mathbb{P}(W_2) \times \cdots \times \mathbb{P}(W_n)$.*

Proof. Each element of the maximal torus T is the image of a diagonal matrix $t := \mathrm{diag}(t_1, \ldots, t_n)$ under the surjection $\mathrm{GL}_n \to \mathrm{PGL}_n$. The kernel of the action of T on $\mathbb{P}(W_i)$ are the elements such that $t_i t_j^{-1}$ are equal for all $j \neq i$. Thus the kernel of the action on X is the subgroup of elements with $t_i = t_j$ for all i, j, so T acts faithfully on X. For generic $x \in X$, the identity component of $N_G(T)_x$ is contained in T_x, so $\mathrm{Lie}(N_G(T)_x) \subseteq \mathrm{Lie}(T)_x = 0$, i.e., $N_G(T)_x$ is finite étale.

To show that the (concrete) group S of k-points of $N_G(T)_x$ is trivial, it suffices to check $1 \neq s \in S$ that

$$\dim s^T + \dim X^s < \dim X \tag{6.2}$$

(compare, for example, [16, 10.2, 10.5]). As $s \neq 1$, it permutes the W_i's non-trivially. If s moves more than two of the W_i's, then

$$\dim X - \dim X^s \geq 2 \dim \mathbb{P}(W_i) = 2(n-2).$$

But of course $\dim s^T \leq n-1$, verifying (6.2) for $n \geq 4$.

If s interchanges only two of the W_i's, i.e., it is a transposition, then $\dim X - \dim X^s = n-2$, but $\dim s^T = 1 < n-2$, and again (6.2) has been verified. ☐

Example 6.3. $\mathrm{ed}(\mathrm{PGL}_2) = 2$, regardless of char k, so type A_1 is a genuine exception to Theorems 1.1 and 1.4 (as $\dim \mathrm{PGL}_2 - 2(\mathrm{rank}\, \mathrm{PGL}_2) - 1 = 0$). Indeed, $H^1(k, G)$ classifies quaternion algebras over k, i.e., the subgroup $\mathbb{Z}/2 \times \mu_2$ of PGL_2 gives a surjection in flat cohomology $H^1(k, \mathbb{Z}/2) \times H^1(k, \mu_2) \to H^1(k, \mathrm{PGL}_2)$, so $\mathrm{ed}(\mathrm{PGL}_2) \leq 2$. On the other hand, the connecting homomorphism $H^1(K, \mathrm{PGL}_2) \to H^2(K, \mu_2)$, which sends a quaternion algebra to its class in the 2-torsion of the Brauer group of K, is nonzero for some extension K, and therefore also $\mathrm{ed}(\mathrm{PGL}_2) \geq 2$.

Entirely parallel comments apply to PGL_3, in which case the surjectivity $H^1(k, \mathbb{Z}/3) \times H^1(k, \mu_3) \to H^1(k, \mathrm{PGL}_3)$ is due to Wedderburn [23, 19.2]. Thus $\mathrm{ed}(\mathrm{PGL}_3) = 2$ and PGL_3 is a genuine exception to Theorem 1.4.

The proof of Theorem 1.4 requires a couple more lemmas.

Lemma 6.4. *Suppose $1 \to A \to B \to C \to 1$ is an exact sequence of group schemes over k. If $H^1(K, C) = 0$ for every $K \supseteq k$, then $\mathrm{ed}(B) \leq \mathrm{ed}(A) \leq \mathrm{ed}(B) + \dim C$.*

Proof. For every K, the sequence

$$C(K) \to H^1(K, A) \to H^1(K, B) \to 1 \tag{6.5}$$

is exact. From here the argument is standard. The surjectivity of the middle arrow gives the first inequality. For the second, take $\alpha \in H^1(K, A)$. There is a field K_0 lying between k and K such that $\operatorname{trdeg}_k K_0 \leq \operatorname{ed}(B)$ and an element $\alpha_0 \in H^1(K_0, A)$ whose image in $H^1(K, B)$ agrees with that of α. Thus, there is a $c \in C(K)$ such that $c \cdot \operatorname{res}_{K/K_0}(\alpha_0) = \alpha$. There is a field K_1 lying between k and K such that $\operatorname{trdeg}_k K_1 \leq \dim C$ such that c belongs to $C(K_1) \subseteq C(K)$. In summary,

$$\operatorname{ed}(\alpha) \leq \operatorname{trdeg}_k(K_1 K_0) \leq \operatorname{trdeg}_k K_1 + \operatorname{trdeg}_k K_0 \leq \operatorname{ed}(B) + \dim C.$$

As this holds for every K and every $\alpha \in H^1(K, A)$, the conclusion follows. ☐

Lemma 6.4 applies, for example, to $A = \operatorname{SL}_n / \mu_m$, $B = \operatorname{GL}_n / \mu_m$, and $C = \mathbb{G}_m$. In that case, one can tease out whether $\operatorname{ed}(\operatorname{SL}_n / \mu_m) = \operatorname{ed}(\operatorname{GL}_n / \mu_m)$ or $\operatorname{ed}(\operatorname{GL}_n / \mu_m) + 1$ by arguing as in [11].

Lemma 6.6. *Suppose m divides $n \geq 2$. Then*

$$\operatorname{ed}(\operatorname{GL}_n / \mu_m) \leq \operatorname{ed}(\operatorname{PGL}_n) + n/m - 1.$$

We omit the proof, which is the same as that for [4, Lemma 7.1] apart from cosmetic details.

Proof of Theorem 1.4. In view of Lemmas 6.1 and 3.1, we find that

$$\operatorname{ed}(\operatorname{PGL}_n) \leq \operatorname{ed}(N_G(T)) \leq \dim X - \dim N_G(T) = n^2 - 3n + 1.$$

Therefore Lemma 6.6 gives

$$\operatorname{ed}(\operatorname{GL}_n / \mu_m) \leq n^2 - 3n + n/m, \tag{6.7}$$

and Lemma 6.4 gives the required bound on $\operatorname{ed}(\operatorname{SL}_n / \mu_m)$. ☐

Lemma 6.8. *Suppose m divides n, and write $n = n'q$ where n' and m have the same prime factors and $\gcd(n', q) = 1$. Then $H^1(K, \operatorname{GL}_n / \mu_m) = H^1(K, \operatorname{GL}_{n'} / \mu_m)$ for every extension K of k and $\operatorname{ed}(\operatorname{GL}_n / \mu_m) = \operatorname{ed}(\operatorname{GL}_{n'} / \mu_m)$.*

Proof. The set $H^1(K, \operatorname{GL}_n / \mu_m)$ is in bijection with the isomorphism classes of central simple K-algebras A of degree n and exponent dividing m. As n' and q are coprime, every such algebra can be written uniquely as $A' \otimes B$ where A' has degree n' and B has degree q [19, 4.5.16]. However, B is split as its exponent must divide $\gcd(q, \exp A)$, i.e., $A \cong M_q(A')$. That is, $H^1(K, \operatorname{GL}_n / \mu_m) = H^1(K, \operatorname{GL}_{n'} / \mu_m)$. As this holds for every extension K of k, the claim on essential dimension follows. ☐

Remark 6.9. One can eliminate m from the bound appearing in Theorem 1.4 to obtain

$$\operatorname{ed}(\operatorname{SL}_n / \mu_m) \leq n^2 - 3n + 1 + n/4 \quad \text{for } m \text{ dividing } n \geq 4.$$

To check this, assume $m < 4$. If $m = 1$, $\operatorname{ed}(\operatorname{SL}_n) = 0$. If $m = 2$, then Theorem 1.3 gives a stronger bound.

If $m = 3$, then write $n = n'q$ for $n' = 3^a$ for some $a \geq 1$ as in Lemma 6.8. If $a = 1$, then $n \geq 6$ and $\mathrm{ed}(\mathrm{GL}_n/\mu_3) = \mathrm{ed}(\mathrm{PGL}_3) = 2$ by Lemma 6.8, which is less than $n^2 - 3n + n/4$. If $a > 1$, then $\mathrm{ed}(\mathrm{GL}_n/\mu_3) \leq \mathrm{ed}(\mathrm{PGL}_{n'}) + n'/3 - 1$; as n' is odd and ≥ 9, [27] gives $\mathrm{ed}(\mathrm{PGL}_{n'}) \leq \frac{1}{2}(n'-1)(n'-2)$, whence the claim.

Remark 6.10. Here is another way to obtain an upper bound on $\mathrm{ed}(\mathrm{SL}_n/\mu_m)$; it is amusing because it requires $\mathrm{char}\, k = p$ to be nonzero. Fix an integer $e \geq 1$ and $\varepsilon = \pm 1$, and set $m := \gcd(p^e + \varepsilon, n)$. We will show that

$$\mathrm{ed}(\mathrm{SL}_n/\mu_m) \leq n^2 - n + 1. \tag{6.11}$$

To see this, consider the GL_n-module $V := W \otimes W^{[e]}$ or $W^* \otimes W^{[e]}$, where W is the natural module k^n, $[e]$ denotes the e-th Frobenius twist, and where we take the first option if $\varepsilon = +1$ and the second option if $\varepsilon = -1$. A scalar matrix $x \in \mathrm{GL}_n$ acts on V as $x^{p^e+\varepsilon}$, and therefore the action of SL_n on V gives a faithful representation of $G := \mathrm{SL}_n/\mu_m$. We consider the action of $N_G(T)$ on V for T a maximal torus in G, and apply Lemma 2.3 to see that $N_G(T)$ acts generically freely on V and so obtain (6.11).

7. Minuscule representations of E_6^{sc} and E_7^{sc}: proof of Theorem 1.5. The smallest nontrivial irreducible representations of $G = E_6^{\mathrm{sc}}$ or E_7^{sc} over $\mathbb{C}$ have dimension 27 or 56 respectively. For each of these, the weights are a single orbit Ω under the Weyl group and occur with multiplicity 1. (The highest weight is minuscule in the sense of [6, §VI.1, Exercises 23, 24].) Put V for the Weyl module of G with the same highest weight over the field k, as defined in [22]. It has the same weights as the corresponding module over $\mathbb{C}$ (and in particular is irreducible).

Proposition 7.1. *Let T be a maximal torus in a simply connected group G of type E_6^{sc} or E_7^{sc} over an algebraically closed field k. Then $N_G(T)$ acts generically freely on V.*

Proof. We apply Lemma 2.3. The map $G \to \mathrm{GL}(V)$ is injective, so T acts faithfully on V. It suffices to verify Lemma 2.3(2).

One can list explicitly the weights Ω of V and find $X = \{\chi_1, \ldots, \chi_6\} \subset \Omega$ with $\sum \chi_i = 0$ and $\chi_i \neq \pm\chi_j$ for $i \neq j$. It suffices, therefore, to check for every minimal normal subgroup H of the Weyl group not containing -1, that $HX \neq X$. For this, it is enough to observe that H has no fixed lines on the vector space $\mathbb{C}[\Phi]$ generated by the roots Φ (because $\mathbb{Z}[\Omega] = \mathbb{Z}[\Phi]$, so H fixes no element of Ω) and that H has no orbits of size $2, 3, \ldots, 6$ (because its maximal subgroups have index greater than 6).

For E_6, H has order 25920 with largest maximal subgroups of index 27. For E_7, H is isomorphic to $\mathrm{Sp}_6(\mathbb{F}_2)$ with largest maximal subgroups of index 28. The description of these Weyl groups from [6, Ch. IV, §4, Exercises 2 and 3] make it obvious that H does not preserve any line in $\mathbb{C}[\Phi]$. $\square$

Proof of Theorem 1.5. The group F_4 has 24 short roots, so by Proposition 4.1, we have

$$\mathrm{ed}(F_4) \leq \mathrm{ed}(N_G(T)) \leq 24 - 4 - 1 = 19.$$

For E_7^{sc}, we apply instead Proposition 7.1 to obtain the desired upper bound.

The group E_6^{sc} has a subgroup $F_4 \times \mu_3$ such that the map in cohomology $H^1(K, F_4 \times \mu_3) \to H^1(K, E_6^{\mathrm{sc}})$ is surjective for every extension $K \supseteq k$, see [15, 9.12], hence $\mathrm{ed}(E_6^{\mathrm{sc}}) \leq \mathrm{ed}(F_4) + 1$. □

8. Proof of Theorem 1.1.

Proof. Suppose first that G has type A_{n-1}, i.e., $G \cong \mathrm{SL}_n / \mu_m$. Assume $m > 1$ for otherwise $\mathrm{ed}(G) = 0$. It is claimed that $\mathrm{ed}(G) \leq n^2 - 2n$. As $\mathrm{ed}(\mathrm{PGL}_3) = 2$, we may assume $n \geq 4$. Combining Theorem 1.4 with the fact that $1 + n/m \leq n$ gives the claim.

Now suppose that G is adjoint. If G is one of the types covered by Proposition 4.1, then we are done by combining that proposition with the inequality $\mathrm{ed}(G) \leq \mathrm{ed}(N_G(T))$. Type B was already addressed in the Introduction. For type G_2, the essential dimension is 3 because $H^1(K, G_2)$ is in bijection with the set of 3-Pfister quadratic forms over K for every field K containing k [23, 26.19].

Now suppose that G is neither type A nor adjoint. If G has type B, then G is a spin group, so there is nothing to prove. If G has type C, then $G = \mathrm{Sp}_{2n}$ and $\mathrm{ed}(G) = 0$. If G has type D, then the only remaining case to consider is $G = \mathrm{SO}_{2n}$ for $n \geq 4$ and then $\mathrm{ed}(G) \leq 2n - 1 < 2n^2 - 3n - 1 = \dim G - 2(\mathrm{rank}\, G) - 1$. The two remaining cases are the simply connected groups of type E_6 and E_7 for which we refer to Theorem 1.5. □

9. Generic stabilizer for the adjoint action. As a complement to the above results, we now calculate the stabilizer in a simple algebraic group G of a generic element in $\mathrm{Lie}(\mathrm{Ad}(G))$. (Note that, in case $G = \mathrm{SL}_2$, we are discussing the action on $\mathrm{Lie}(\mathrm{PGL}_2)$, not on $\mathrm{Lie}(\mathrm{SL}_2)$, and the two Lie algebras are distinct if $\mathrm{char}\, k = 2$.) We include this calculation here because the methods are similar to the previous results. The results are complementary, in the sense that previously we considered $N_G(T)$ acting on representations with no zero weights, and in this section we consider $N_G(T)$ acting on $\mathrm{Lie}(\mathrm{Ad}(T))$, for which zero is the only weight. The main result, Proposition 9.2, is used in [17].

After a preliminary result, we will calculate the stabilizer of a generic element of the adjoint representation. Let Φ be an irreducible root system, and put W for its Weyl group and Q for its root lattice. For each prime p, tensoring Q with the finite field $\mathbb{F}_p$ gives a homomorphism

$$\rho_p \colon \langle W, -1 \rangle \to \mathrm{GL}_{\mathrm{rank}\, Q}(\mathbb{F}_p).$$

Lemma 9.1. *The kernel of ρ_p is $(\mathbb{Z}/2)^n$ if Φ has type B_n for some $n \geq 2$ and $p = 2$. Otherwise,* $\ker \rho_p = \langle -1 \rangle$ *if $p = 2$ and* $\ker \rho_p = 1$ *for $p \neq 2$.*

Proof. If $p \neq 2$, $\ker \rho_p = 1$ by an old theorem of Minkowski (see [35] and also [43, Lemma 1.1]). So we may assume that $p = 2$. It also follows by a similar argument that $\ker \rho_2$ is a 2-group [43, Lemma 1.1']. Clearly $-1 \in \ker \rho_2$. Thus, the result follows immediately for G of type A_n for $n \neq 3$, G_2, or E_n since the only normal 2-subgroups in these cases are the subgroup of order 2 containing -1. It is straightforward to check the result for the groups $A_3 = D_3$ and C_3.

Note that the root lattice of D_{n-1} is a direct summand of $D_n, n > 3$, and any normal 2-subgroup of the Weyl group of D_n of order greater than 2 intersects the Weyl group of D_{n-1} in a subgroup of order greater than 2. Thus, the result for D_3 implies the result for all D_n. Similarly, the result for C_3 implies the result for $C_n, n > 3$.

Finally, suppose Φ has type B_n for some $n \geq 2$ and $p = 2$. Viewing $\mathbb{Z}^n$ as having basis ε_i for $1 \leq i \leq n$, we can embed Φ in $\mathbb{Z}^n$ by setting the simple roots to be $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i < n$ and $\alpha_n = \varepsilon_n$ as in [6]. The Weyl group W is isomorphic to $(\mathbb{Z}/2)^n \rtimes S_n$, where $(\mathbb{Z}/2)^n$ consists of all possible sign flips of the ε_i and S_n acts by permuting the ε_i. The subgroup $(\mathbb{Z}/2)^n$ obviously acts trivially on $Q \otimes \mathbb{F}_2$ (since there is a basis of eigenvectors for Q for this subgroup of exponent 2). In fact, $(\mathbb{Z}/2)^n$ is precisely the kernel of the action of W on $Q \otimes \mathbb{F}_2$, as it is easy to check for $n < 5$ and is clear for $n \geq 5$. □

Proposition 9.2. *Let G be a simple algebraic group. The action of G on* Lie $(\mathrm{Ad}(G))$ *has stabilizer in general position S, with identity component S° a maximal torus in G. Moreover, $S = S^\circ$ unless* $\operatorname{char} k = 2$ *and*

1. *G has type B_n for $n \geq 2$; in this case $S/S^\circ \cong (\mathbb{Z}/2)^n$.*
2. *G has type A_1, C_n for $n \geq 3$, D_n for $n \geq 4$ even, E_7, E_8, F_4, or G_2; in this case $S/S^\circ \cong \mathbb{Z}/2$ and the nontrivial element acts on S° by inversion.*

Proof. Suppose first that $G = \mathrm{Ad}(G)$ and fix a maximal torus T of G. As G is adjoint, the Lie algebra $\mathrm{Lie}(T)$ is a Cartan subalgebra of $\mathrm{Lie}(G)$, and the natural map $G \times \mathrm{Lie}(T) \to \mathrm{Lie}(G)$ is dominant [13, XIII.5.1, XIV.3.18]. Therefore, it suffices to verify that the stabilizer S in G of a generic vector t in $\mathrm{Lie}(T)$ is as claimed. The subgroup of G transporting t in $\mathrm{Lie}(T)$ is the normalizer $N_G(T)$ [13, XIII.6.1(d)(viii)], hence S is the centralizer of t in $N_G(T)$, and it follows that $S^\circ = T$ and S/S° is isomorphic to the group of elements w of the Weyl group fixing t, compare [44, Lemma 3.7]. As G is adjoint, the element t is determined by its action on $\mathrm{Lie}(G)$, i.e., by the values of the roots on t; in particular, $w(t) = t$ if and only if w acts trivially on $Q \otimes k$. Lemma 9.1 completes the proof for G adjoint.

In case G is not adjoint, the representation factors through the central isogeny $G \to \mathrm{Ad}(G)$, and G_t is the inverse image of the generic stabilizer in $\mathrm{Ad}(G)$. □

To summarize the proof, the identity component of $C_G(t)$ is T by [13], so $C_G(t)$ is contained in $N_G(T)$ and is determined by its image in the Weyl group $N_G(T)/T$ this statement is included in [44]. What is added here is the calculation of the component group $C_G(t)/T$, and in particular that it need not be connected.

One can also compute the generic stabilizer for the action of G on the projective space $\mathbb{P}(\mathrm{Lie}(G))$ of $\mathrm{Lie}(G)$ by the same argument. If $p = 2$, since $\mathrm{PGL}_n(\mathbb{F}_2) = \mathrm{GL}_n(\mathbb{F}_2)$, we see that the generic stabilizers for $\mathbb{P}(\mathrm{Lie}(G))$ and $\mathrm{Lie}(G)$ are the same. If p is odd, an easy argument shows that a generic stabilizer is a maximal torus if -1 is not in the Weyl group and is just a maximal torus extended by -1 if -1 is in the Weyl group. (Clearly, -1 does act by

-1 on $\mathrm{Lie}(T)$, T a maximal torus.) In any case, the connected component of the stabilizer of a generic line in $\mathrm{Lie}(G)$ is contained in the normalizer of a maximal torus, as we know from [13].

Corollary 9.3. *Suppose V is a faithful representation of an algebraic group G over an algebraically closed field k. If*

1. *G is connected reductive and* $\mathrm{char}\, k \neq 2$ *or*
2. *G is simple of type B_n, C_n, or D_n,*

then $\mathrm{ed}(G) \leq \dim V$*.*

If one assumes that $\mathrm{char}\, k = 0$, then the conclusion holds with milder hypotheses on G. This follows easily from the main result in [10], see [8, Lemma 13.9]. We use an entirely different argument.

Proof. Pick a maximal torus T in G. If $\mathrm{char}\, k \neq 2$, put $\mathfrak{t}$ for the Lie algebra of the image of T in $\mathrm{Ad}(G)$. By Lemma 9.1, the stabilizer in $N_G(T)$ of a generic element of $\mathfrak{t}$ is T. Thus, $N_G(T)$ acts generically freely on $V \oplus \mathfrak{t}$ and the claim on $\mathrm{ed}(G)$ follows because $\dim \mathfrak{t} \leq \dim T$.

If $\mathrm{char}\, k = 2$ and G has type B_n with $n \geq 2$, the Weyl group $W := N_G(T)/T$ is a semi-direct of $(\mathbb{Z}/2)^n$ and S_n. In particular, W embeds in $\mathrm{AGL}_n(\mathbb{F}_2)$, the $\mathbb{F}_2$-points of the group of affine transformations of $\mathbb{A}^n_{\mathbb{F}_2}$. Clearly, AGL_n embeds in $\mathrm{GL}_{n+1} = \mathrm{PGL}_{n+1}$. Thus, W acts faithfully, hence generically freely on $\mathbb{P}^n$, $N_G(T)$ acts generically freely on $V \oplus \mathbb{P}^n$, and the claim on $\mathrm{ed}(G)$ follows from Lemma 3.1. Since the Weyl groups of C_n and D_n equal or embed in the Weyl group of B_n, the same is true for those groups. For type $B_1 = A_1$, S_2 injects into $\mathrm{GL}_2(\mathbb{F}_2) = \mathrm{PGL}_2(\mathbb{F}_2)$. $\square$

While the corollary gives cheap upper bounds on $\mathrm{ed}(G)$, it is not sufficient to deduce Theorem 1.1 even in the coarse sense of big-O notation and in good characteristic: the minimal faithful representations of SL_n/μ_m are too big for $3 \leq m < n$, being at least cubic in n whereas the dimension of the group is $n^2 - 1$.

The hypotheses in the corollary could be relaxed, but we ignore this because the rest of this article already proves stronger bounds on $\mathrm{ed}(G)$. We include (1) here because it follows so easily from the results of this section, and we include the case (2) because we will use it for spin and half-spin groups in the next section.

Action of G on $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$. In case $k = \mathbb{C}$, it is well-known that an adjoint simple group G acts generically freely on $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$. However, we have also the following:

Example 9.4. Maintaining the notation of Example 4.2, the Lie algebra $\mathfrak{pgl}_2$ of PGL_2 may be identified with the Lie algebra $\mathfrak{gl}_2$ of 2-by-2 matrices, modulo the scalar matrices. Write T for the (image of the) diagonal matrices in PGL_2. A generic element $v \in \mathfrak{pgl}_2$ is the image of some $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$. The normalizer of $[v] \in \mathbb{P}(\mathfrak{pgl}_2)$ in $N_G(T)$ is $\mathbb{Z}/2$, with nontrivial element the image g of $\left(\begin{smallmatrix} 0 & y \\ -z & 0 \end{smallmatrix}\right)$, which satisfies $gv = -v$. If $\mathrm{char}\, k = 2$, the same calculation shows that the normalizer of $v \in \mathfrak{pgl}_2$ is $\mathbb{Z}/2$.

The subgroup of PGL_2 mapping a generic element of $\mathrm{Lie}(T)$ into $\mathrm{Lie}(T)$ is $N_G(T)$, as was already used in the proof of Proposition 9.2. Therefore, the stabilizer in G of a generic element of $\mathbb{P}(\mathfrak{pgl}_2) \oplus \mathbb{P}(\mathfrak{pgl}_2)$ equals the stabilizer in $N_G(T)$ of a generic element of $\mathbb{P}(\mathfrak{pgl}_2)$, i.e., $\mathbb{Z}/2$.

Moreover, if $\mathrm{char}\, k = 2$, *the stabilizer in* PGL_2 *of a generic element in* $\mathfrak{pgl}_2 \oplus \mathfrak{pgl}_2$ *is* $\mathbb{Z}/2$.

We note that this is the only such example.

Proposition 9.5. *Let G be an adjoint simple group. Then G acts generically freely on $\mathbb{P}(\mathrm{Lie}(G)) \times \mathbb{P}(\mathrm{Lie}(G))$ unless G has type A_1. If G has type A_1 and* $\mathrm{char}\, k \neq 2$, *then G acts generically freely on* $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$.

Proof. Pick a maximal torus T in G. The stabilizer in G of a generic element of $\mathbb{P}(\mathrm{Lie}(G)) \times \mathbb{P}(\mathrm{Lie}(G))$ is contained in the intersection of two generic conjugates of $N_G(T)$. If G is not of type A_1, then this intersection is 1 as in the proof of [7, Cor. 10]. If G is of type A_1 and $\mathrm{char}\, k \neq 2$, then we apply the preceding example. $\square$

Note that if $p \neq 2$ and we consider the action of G on $\mathrm{Lie}(G)$, then a generic stabilizer is a maximal torus and it is elementary to see that two generic conjugates of a maximal torus intersect trivially.

10. Groups of type *B* and *D*. We have not yet discussed upper bounds for the simply connected groups Spin_n for $n \geq 7$ of type B_ℓ for $\ell \geq 3$ or D_ℓ for $\ell \geq 4$. Also, for Spin_n with n divisible by 4 and at least 12, there is a quotient Spin_n / μ_2 that is distinct from SO_n; it is denoted HSpin_n and is known as a half-spin group.

The group $G = \mathrm{Spin}_n$ with $n > 14$ or HSpin_n with $n > 16$ acts generically freely on a (half) spin representation or the sum of a half spin representation and the vector representation $\mathrm{Spin}_n \to \mathrm{SO}_n$ by [2,36] if $\mathrm{char}\, k = 0$ and [17] for all characteristics. This gives an upper bound on $\mathrm{ed}(G)$, which is an equality if $\mathrm{char}\, k \neq 2$, see [9,17].

We now give bounds for HSpin_{12} and HSpin_{16}.

Lemma 10.1. *For T a maximal torus in $G := \mathrm{HSpin}_n$ for n divisible by 4 and $n \geq 12$, the group $N_G(T)$ acts generically freely on the half-spin representation of G.*

Proof. Apply Lemma 2.3. The representation V is minuscule and T acts faithfully because G does so. The element -1 of the Weyl group acts nontrivially on $\ker \psi$ by Example 2.5 because $\frac{1}{2} \dim V = 2^{n/2-2} > n/2 = \dim T$. As -1 is contained in every nontrivial normal subgroup of the Weyl group, the proof is complete. $\square$

Corollary 10.2. *Over every algebraically closed field,*

$$\mathrm{ed}(\mathrm{HSpin}_{12}) \leq 26 \quad \textit{and} \quad \mathrm{ed}(\mathrm{HSpin}_{16}) \leq 120.$$

The remaining groups are Spin_n with $7 \leq n \leq 14$. In case $\mathrm{char}\, k \neq 2$, the precise essential dimension is known by Rost, see [15,40,41]. The same methods, combined with the calculations of the generic stabilizers from [17], will provide upper bounds for $\mathrm{ed}(\mathrm{Spin}_n)$ in case $\mathrm{char}\, k = 2$. But these methods require detailed arguments, so for our purposes we note simply that Spin_n acts faithfully on the spin representation for n odd and on the direct sum of the vector representation and a half-spin representation for n even; Corollary 9.3 then provides an upper bound on $\mathrm{ed}(\mathrm{Spin}_n)$. This completes the task of giving an upper bound on $\mathrm{ed}(G)$ for every simple algebraic group G over an algebraically closed field k.

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Skip Garibaldi
Center for Communications Research,
San Diego,
CA 92121,
USA
e-mail: `skip@member.ams.org`

Robert M. Guralnick
Department of Mathematics,
University of Southern California,
Los Angeles,
CA 90089-2532,
USA
e-mail: `guralnic@usc.edu`

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