

# Algebraic groups with few subgroups

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## ABSTRACT

Every semisimple linear algebraic group over a field  $F$  contains nontrivial connected subgroups, namely maximal tori. In the early 1990s, J. Tits proved that some groups of type  $E_8$  have no others. We give a simpler proof of his result, prove that some groups of type  ${}^3D_4$  and  ${}^6D_4$  have no nontrivial connected subgroups, and give partial results for types  $E_6$  and  $E_7$ . Our result for  ${}^3D_4$  uses a general theorem on the indexes of Tits algebras which is of independent interest.

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## 1. Introduction

In [Ti 92b], Jacques Tits proved that there exist algebraic groups of type  $E_8$  that have very few subgroups. Specifically, he proved that they have no proper, nontrivial closed connected subgroups other than maximal tori. Tits also gave a partial result towards the existence of groups of type  $E_6$  with the same property in [Ti 92a].

We say that a reductive (connected) group  $G$  defined over a field  $F$  is *almost abelian*<sup>†</sup> if  $G/F$  has no proper, nontrivial reductive  $F$ -subgroups other than maximal tori. In characteristic zero, an almost abelian reductive  $F$ -group  $G$  has no proper, nontrivial closed connected subgroups other than maximal tori, see Lemma 2.4. In this language, Tits proved that some algebraic groups of type  $E_8$  over some fields of characteristic zero are almost abelian.

Since his work, new tools have appeared, making the problem of construction almost abelian groups easier. We give a simpler proof of his result for groups of type  $E_8$  (§9), extend his result

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2000 *Mathematics Subject Classification* 20G15.

The first author's research was partially supported by National Science Foundation grant DMS-0653502. He thanks the Institut des Hautes Etudes Scientifiques for its hospitality while the work on this article was done.

<sup>†</sup>This term was suggested by A. Premet.

for groups of type  $E_6$  (Propositions 11.1 and 12.1), and prove a similar partial result for groups of type  $E_7$  (Proposition 13.1). Roughly speaking, for groups of types  ${}^1E_6$ ,  ${}^2E_6$  and  $E_7$ , we settle the question of existence of reductive subgroups up to prime-to- $p$  extensions of the base field for each prime  $p$ .

For groups of type  $D_4$ , we can do more. We prove:

**THEOREM 1.1.** *If  $G$  is a superversal group of type  ${}^3D_4$  or  ${}^6D_4$  over a field of characteristic  $\neq 3$ , then  $G$  is almost abelian.*

This answers a question by A. Premet and is proved in §15. The proof uses a result about maximal indexes of Tits algebras for groups of outer type (Th. 14.1), which is interesting by itself.

A superversal group is roughly speaking the most general group of the given type, see Definition 6.3 below. For global fields, such examples do not occur, see section 5.

We remark that groups of type  $G_2$  or  $F_4$  are never almost abelian, because the long roots generate a subgroup of type  $A_2$  or  $D_4$  respectively. Thus we have treated the question of existence of reductive subgroups for all exceptional groups, except that for types  $E_6$  and  $E_7$  we have only done so up to prime-to- $p$  extensions of the base field for each prime  $p$ .

The idea behind Tits's proofs and ours is to find groups that require a field extension of large degree to split them. We combine his methods with the recent computations of torsion indexes by Totaro [To] and the Rost invariant defined in [Mer 03].

### Notation and conventions

The groups we discuss are affine and algebraic, in particular an  $F$ -subgroup  $H \subset G$  is always closed [SGA3, VI<sub>B</sub>.1.4.2]. Mainly we deal with smooth affine algebraic groups which are exactly the linear algebraic groups in the sense of [Bo]. We write  $\mu_n$  for the group scheme of  $n$ -th roots of unity. We define "reductive group" to include the hypothesis that the group is connected. We write  $G^0$  for the identity component of  $G$ . A group is *trivial* if it is 1 and *nontrivial* otherwise.

For a field  $F$ , we write  $\text{Gal}(F)$  for its absolute Galois group, i.e., the Galois group of a separable closure  $F_{\text{sep}}$  of  $F$  over  $F$ . If  $\text{Gal}(F)$  is a pro- $p$ -group (i.e., if every finite separable extension of  $F$  is of degree a power of  $p$ ), then we say that  $F$  is  *$p$ -special*. The subfield  $F_p$  of  $F_{\text{sep}}$  consisting of elements fixed by a  $p$ -Sylow subgroup of  $\text{Gal}(F)$  is  *$p$ -special*. We call  $F_p$  a *co- $p$ -closure* of  $F$ .

For a central simple  $F$ -algebra  $D$ , we write  $\text{SL}_1(D)$  for the algebraic groups whose  $F$ -points are the norm 1 elements of  $D$ .

We use extensively the notion of Tits index of a semisimple group as defined in [Ti 66]. The list of possible indexes is given in that paper and also in [Sp, pp. 320, 321]. (There is a typo in index #14 in Springer's table; it needs to have an additional circle darkened. His Proposition 17.8.2 is correct.)

## 2. First steps

Let  $G$  be a reductive algebraic group over a field  $F$ . By the *type* of  $G$ , we mean its Killing-Cartan type over an algebraic closure of  $F$ .

2.1. The property of being almost abelian is an invariant of isogeny classes. That is, for every central isogeny  $G \rightarrow G'$ , the group  $G$  is almost abelian if and only if  $G'$  is.

2.2. Our general plan is the following. For a specific semisimple group  $G$  over a field  $F$ , we prove two statements:

(1)  $G$  does not contain any  $F$ -defined proper semisimple subgroups of the same absolute rank as  $G$ .

(2) Every nontrivial  $F$ -torus in  $G$  is maximal.

This will show that every nontrivial proper (connected) reductive  $F$ -subgroup of  $G$  is a maximal torus. Indeed, let  $H$  be a nontrivial proper  $F$ -reductive subgroup of  $G$ . By (2),  $H$  has the same absolute rank as  $G$ . Again by (2), the central torus of  $H$  is either trivial or all of  $H$ , i.e.,  $H$  is semisimple (excluded by (1)) or a maximal torus, as claimed.

In order to prove (1), we note that an  $F$ -defined semisimple subgroup  $H$  of  $G$  of the same absolute rank contains—and so is normalized by—a maximal torus of  $G$ , hence  $H$  is generated over an algebraic closure of  $F$  by root subgroups of  $G$  [Bo, 13.20]. That is, the roots of  $H$  are a subsystem of the roots of  $G$ , and  $H$  is of one of the types described in [BdS] or [Dy, Th. 5.3], even though we do not assume that  $F$  has characteristic zero.

Item (2) is the difficult one to prove, in practice. For proving it, our work is cut in half by the following observation whose proof amounts to the fact that the category of finite-dimensional representations of a finite group over  $\mathbb{Q}$  is semisimple.

**SYMMETRY LEMMA 2.3.** *Let  $G$  be a reductive algebraic group of absolute rank  $n$ . If  $G$  contains an  $F$ -torus of rank  $m$ , then it contains an  $F$ -torus of rank  $n - m$ .*

*Proof.* Let  $S$  be an  $F$ -torus in  $G$  of rank  $m$ . It is contained in a maximal  $F$ -torus  $T$  of rank  $n$ . We switch to the dual category of finitely generated abelian groups with a continuous action by  $\text{Gal}(F)$ ; by continuity, the action of  $\text{Gal}(F)$  on  $T^*$  and  $S^*$  factors through a finite group. Semisimplicity implies that the natural projection  $T^* \rightarrow S^*$  splits over  $\mathbb{Q}$ , i.e., there is a map  $f: S^* \otimes \mathbb{Q} \rightarrow T^* \otimes \mathbb{Q}$  that is compatible with the Galois action and such that  $f(\beta)|_S = \beta$  for all  $\beta \in S^*$ . Set  $S'_0$  to be the intersection of  $f(S^* \otimes \mathbb{Q})$  with  $T^*$  and

$$S' := \left( \bigcap_{\alpha \in S'_0} \ker \alpha \right)^0.$$

Then  $S'$  is an  $F$ -subtorus of  $T$  such that  $T = S.S'$  and  $S \cap S'$  is finite; it is the desired torus.  $\square$

A torus over  $F$  is  $F$ -irreducible if it has no nontrivial proper subtori defined over  $F$ .

**LEMMA 2.4.** *Let  $G/F$  be an almost abelian reductive group.*

- (1)  $G$  is an  $F$ -irreducible torus or  $G$  is semisimple.
- (2)  $G = \mathbb{G}_m$  or  $G$  is  $F$ -anisotropic.
- (3) If  $F$  is perfect, then  $G$  has no proper, nontrivial smooth connected  $F$ -subgroups other than maximal tori.
- (4) If  $\text{char}(F) = 0$ , then  $G$  has no proper, nontrivial connected subgroups other than maximal tori.

*Proof.* (1) The connected center  $Z(G)^0$  is an  $F$ -subtorus of  $G$ . Hence  $Z(G)^0 = G$  or  $Z(G)^0 = 1$ , so  $G$  is a torus or  $G$  is semisimple. If  $G$  is an almost abelian torus, we have already seen that  $G$  is irreducible.

(2) Assume that  $G$  is isotropic, i.e., there exists an  $F$ -embedding  $\mathbb{G}_m \rightarrow G$ . Then we have an isomorphism  $\mathbb{G}_m \cong G$ .

(3) Let  $H$  be a smooth connected  $F$ -subgroup of  $G$  and consider its unipotent radical  $U = R_u(H)$  [Sp, §12.1.7] which is a smooth connected unipotent subgroup of  $H$  and defined over  $F$  (because  $F$  is perfect). By [BoTi, 3.7],  $U(F_{\text{sep}})$  is a subgroup of the unipotent radical of the

type	torsion primes
$A_n (n \geq 1), C_n (n \geq 2)$	none
$B_n (n \geq 3), D_n (n \geq 4), G_2$	2
$F_4, E_6, E_7$	2, 3
$E_8$	2, 3, 5

TABLE 2. Torsion primes for simple root systems, cf. [St, 1.13]

$F_{\text{sep}}$ -parabolic subgroup  $\mathcal{P}(U(F_{\text{sep}}))$  constructed by Borel-Tits. Furthermore, this parabolic  $F_{\text{sep}}$ -subgroup descends in a  $F$ -parabolic subgroup  $P$  and  $U$  is then a  $F$ -subgroup of  $R_u(P)$ . If  $U$  is nontrivial, then  $P$  is proper and  $G$  is isotropic which contradicts (1). Hence  $H$  is reductive and is a maximal  $F$ -torus.

(4) follows from (3) since all algebraic groups are smooth in characteristic zero (Cartier).  $\square$

2.5. Suppose now that  $G$  is semisimple and simply connected, that  $F$  is of positive characteristic  $p$  which is not a torsion prime for  $G$  as listed in Table 2. Assume that  $G/F$  is almost abelian. Then by [Ti 87, 2.6], the argument of 2.4(3) still shows that  $G$  has no proper connected smooth unipotent  $F$ -subgroups. We can then conclude in this case that the connected smooth  $F$ -subgroups of  $G$  are maximal tori or pseudo-reductives (in the sense of [CGP]). For example, if  $A = (L/k, \sigma, a)$  is a cyclic division algebra of degree  $p$ , the group  $\text{SL}_1(A)$  is almost abelian and contains the pseudoreductive commutative  $k$ -subgroup  $R_{k(\sqrt[p]{a})/k}^1(\mathbb{G}_m)$ .

### 3. Scholium: groups of type $A_1 \times A_1 \times A_1$

We illustrate the material in the preceding section by studying the almost abelian groups of type  $A_1 \times A_1 \times A_1$  over a field  $F$  of characteristic  $\neq 2$ . (This material is not used in the rest of the paper.) Every group of this type is isogenous to a transfer  $R_{L/F}(\text{SL}_1(Q))$ , where  $L$  is a cubic étale  $F$ -algebra and  $Q$  is a quaternion  $L$ -algebra. If  $L$  is not a field, then such a group is obviously not almost abelian, because it contains an  $F$ -defined subgroup of type  $A_1$ . We have:

PROPOSITION 3.1. *Maintain the notation of the previous paragraph and suppose that  $L$  is a field. Then:*

- (1)  $R_{L/F}(\text{SL}_1(Q))$  is almost abelian if and only if  $Q$  does not contain a quadratic étale  $F$ -algebra.
- (2) If  $\text{cor}_{L/F}(Q)$  has index  $\geq 4$ , then  $R_{L/F}(\text{SL}_1(Q))$  is almost abelian.
- (3) If  $R_{L/F}(\text{SL}_1(Q))$  is almost abelian, then  $\text{cor}_{L/F}(Q)$  has index  $\geq 2$ .

*Proof.* Suppose first that  $Q$  does not contain a quadratic étale  $F$ -algebra. The group has no proper semisimple subgroups of rank 3 over an algebraic closure, so it cannot have one over  $F$ . If  $R_{L/F}(\text{SL}_1(Q))$  contains a rank 1  $F$ -torus, then there is a quadratic extension of  $F$  splitting  $Q$ , which is impossible by the assumption on  $Q$ . By the Symmetry Lemma,  $R_{L/F}(\text{SL}_1(Q))$  also has no rank 2 tori. Applying 2.2, we conclude that  $R_{L/F}(\text{SL}_1(Q))$  is almost abelian.

Conversely, suppose that  $Q$  contains a quadratic étale  $F$ -algebra  $K$ . The group  $R_{L/F}(\text{SL}_1(Q))$  contains the maximal torus  $R_{L/F}(R_{K \otimes L/L}^1(\mathbb{G}_m))$ , hence the 1-dimensional torus  $R_{K/F}^1(\mathbb{G}_m)$ . We have proved (1).

$Q$  is  $(a, b)$  for some  $a, b \in L^\times$ , i.e., it is generated as an  $L$ -algebra by elements  $i, j$  such that  $i^2 = a, j^2 = b$ , and  $ij = -ji$ . It contains a quadratic étale  $F$ -algebra if and only if  $b$  can be

chosen to lie in  $F^\times$ . Claim (2) follows from (1): if  $b$  is in  $F^\times$ , then  $\text{cor}_{L/F}(Q)$  is Brauer-equivalent to  $(N_{L/F}(b), a)$ .

If  $\text{cor}_{L/F}(Q)$  is split, then by [KMRT, 43.9],  $Q$  contains a quadratic étale  $F$ -algebra and (3) follows. □

In the case where  $\text{cor}_{L/F}(Q)$  has index 2, the group  $R_{L/F}(\text{SL}_1(Q))$  can be almost abelian or not. Examples where the group is not almost abelian are easy to create using the projection formula as in the proof of part (2) of the proposition.

**EXAMPLE 3.2.** Let's sketch an example where the corestriction has index 2 and the group is almost abelian. Section 4 of the paper [RT] gives an example of a cyclic Galois extension  $L_0/F_0$  of degree 3 and a quaternion algebra  $Q_0$  over  $L_0$  such that  $\text{cor}_{L_0/F_0}(Q_0)$  has index 8. Let  $X$  be the projective variety of right ideals of dimension 16 in  $\text{cor}_{L_0/F_0}(Q_0)$ . Put  $F := F_0(X)$  and  $L := L_0(X)$ . As  $F$  is a regular extension of  $F_0$ ,  $L$  is canonically identified with  $L_0 \otimes_{F_0} F$ . For  $Q := Q_0 \otimes_{L_0} L_0(X)$ , we have

$$\text{cor}_{L/F}(Q) = \text{cor}_{L/F} \text{res}_{L/L_0}(Q_0) = \text{res}_{F/F_0} \text{cor}_{L_0/F_0}(Q_0),$$

so  $\text{cor}_{L/F}(Q)$  has index 2.

For sake of contradiction, suppose that  $R_{L/F}(\text{SL}_1(Q))$  is not almost abelian, so  $Q$  can be written as  $(a, b)$  for some  $b \in F^\times$ . Write  $\rho$  for a generator of the Galois group of  $L/F$ , equivalently,  $L_0/F_0$ . Then  $Q \otimes^\rho Q$  is Brauer-equivalent to  $(a \rho(a), b)$ , hence has index 2. On the other hand,  $Q \otimes^\rho Q$  is  $(Q_0 \otimes^\rho Q_0) \otimes_{L_0} L_0(X)$ . But  $Q_0 \otimes^\rho Q_0$  has index 4 (because  $\text{res}_{L_0/F_0} \text{cor}_{L_0/F_0}(Q_0)$  has index 8), and the index reduction formula from [MPW1, p. 565] shows that  $L_0(X)$  does not lower the index of  $Q_0 \otimes^\rho Q_0$ . This is a contradiction, so  $R_{L/F}(\text{SL}_1(Q))$  is almost abelian.

#### 4. Splitting fields of tori

We recall the well-known divisibility bounds on finite subgroups of  $\text{GL}_n(\mathbb{Z})$  and  $\text{GL}_n(\mathbb{Q})$  given by Table 4. (We remark that every finite subgroup of  $\text{GL}_n(\mathbb{Q})$  is conjugate to a finite subgroup in  $\text{GL}_n(\mathbb{Z})$  by [Se 92, p. 124], so the bounds are the same whether one takes  $\mathbb{Z}$  or  $\mathbb{Q}$ .)

$n$	finite subgroups of $\text{GL}_n(\mathbb{Z})$ or $\text{GL}_n(\mathbb{Q})$ have order dividing	reference
1	2	
2	$2^3$ or $2^2 \cdot 3$	e.g. [KP, 4.1]
3	$2^4 \cdot 3$	[Ta]
4	$2^7 \cdot 3^2$ or $2^3 \cdot 5$ or $2^4 \cdot 3 \cdot 5$	[Da], [Ti 92b, p. 1137]

TABLE 4. Divisibility bound on finite subgroups of  $\text{GL}_n(\mathbb{Z})$  and  $\text{GL}_n(\mathbb{Q})$

The  $F$ -isomorphism class of an  $n$ -dimensional torus  $T$  can be specified by a continuous homomorphism  $\phi: \text{Gal}(F) \rightarrow \text{GL}_n(\mathbb{Z})$  (representing the natural Galois action on  $T^*$ ), and  $T$  is split over  $F$  if and only if  $\phi$  is zero. It follows that every torus of rank  $n$  can be split by a field extension of degree  $|\text{im } \phi|$ , and this order is bounded as in Table 4.

Let us start with the following elementary fact.

PROPOSITION 4.1 (type  ${}^1A_{p-1}$ ). *Let  $D$  be a central division algebra of prime degree  $p$ . Then the group  $\mathrm{SL}_1(D)$  is almost abelian.*

*Proof.* We follow 2.2. The group  $\mathrm{SL}_1(D)$  is clearly anisotropic, and 2.2(1) holds because there are no proper semisimple subgroups of  $\mathrm{SL}_1(D)$  of rank  $p-1$  even over an algebraic closure of  $F$ .

It remains to show that every non-maximal torus in  $\mathrm{SL}_1(D)$  is trivial. First, we may replace  $F$  by a co- $p$ -closure  $F_p$ , because  $D$  remains division over  $F_p$ . The torus  $T$  is then a norm one torus  $R_{K/F}^1(\mathbb{G}_m)$  for a cyclic Galois  $F$ -algebra  $K$  of dimension  $p$ . Since  $T$  is anisotropic,  $K$  is a cyclic field extension of  $F$ . The corresponding representation  $\mathrm{Gal}(K/F) \rightarrow \mathrm{GL}(\mathbb{Z}^{p-1}) \rightarrow \mathrm{GL}(\mathbb{Q}^{p-1})$  is  $\mathbb{Q}$ -irreducible. Thus  $T$  does not admit nontrivial proper subtori.  $\square$

For contrast, we note that a group of type  ${}^1A_{n-1}$  where  $n$  has at least two distinct prime factors is never almost abelian because a central simple algebra of degree  $n$  is decomposable, see e.g. [GiS, 4.5.16].

COROLLARY 4.2 (type  ${}^2A_{p-1}$ ). *Let  $D$  be a central division algebra of prime degree  $p$  over a quadratic extension  $L/F$ . If  $D$  has a unitary involution  $\tau$ , then the group  $\mathrm{SU}(D, \tau)$  is almost abelian.*

*Proof.* Over  $L$ , the group  $\mathrm{SU}(D, \tau)$  becomes isomorphic to  $\mathrm{SL}_1(D)$ , which is almost abelian by Proposition 4.1.  $\square$

Over special fields, other inner groups of type  $A$  are not almost abelian.

LEMMA 4.3 (type  ${}^1A_{p^r-1}$ ). *Assume that  $F$  is  $p$ -special and let  $D$  be an  $F$ -central division algebra of degree  $p^r > p$ . Then the group  $\mathrm{SL}_1(D)$  is not almost abelian.*

*Proof.* Let  $L \subset D$  be a separable maximal subfield of  $D$ , so  $[L : F] = p^r$  (see [GiS, §4.5]). Since  $\mathrm{Gal}(F)$  is a pro- $p$ -group, it follows that there exists an intermediate extension  $F \subset K \subset L$  with  $[K : F] = p$  (*ibid.*, 6.1.8). Thus the torus  $R_{K/F}^1(\mathbb{G}_m)$  is a proper subtorus of the maximal torus  $R_{L/F}^1(\mathbb{G}_m)$  of  $\mathrm{SL}_1(D)$ .  $\square$

EXAMPLE 4.4 (type  ${}^1A_3$ ). By a result of Albert [J, Th. 2.9.21], we know that a central division algebra  $D$  of degree 4 contains a maximal subfield which is Galois of group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . Hence  $\mathrm{SL}_1(D)$  contains a one dimensional torus  $R_{K/k}^1(\mathbb{G}_m)$ . Semisimple groups of type  ${}^1A_3$  are not almost abelian groups.

We ignore whether there exist almost abelian groups of type  ${}^1A_{p^2-1}$  for  $p$  odd.

EXAMPLE 4.5 (types  $B$ ,  $C$ , and  $D$ ). Suppose for convenience that  $F$  has characteristic  $\neq 2$ .

A group of type  $B_n$  for some  $n \geq 2$  is isogenous to the special orthogonal group of a nondegenerate quadratic form  $\sum_{i=1}^{2n+1} a_i x_i^2$  for some  $a_i \in F^\times$  and indeterminates  $x_i$  [KMRT, 26.12]. Such a group is not almost abelian because it contains the special orthogonal group of, say,  $\sum_{i=1}^{2n} a_i x_i^2$  as a subgroup.

A group  $G$  of type  $C_n$  for  $n \geq 3$  (respectively,  ${}^1D_n$  or  ${}^2D_n$  for  $n \geq 4$ ) is isogenous to the symplectic (resp., special orthogonal) group of a central simple algebra  $A$  of degree  $2n$  with involution  $\sigma$  of the first kind, see [KMRT, 26.14, 26.15, 44.7]. If  $A$  is not a division algebra — e.g., if  $n$  is not a power of 2 — then  $G$  is not almost abelian by the same reasoning as in the

preceding paragraph. We ignore the case where  $A$  is division and  $n$  is a power of 2, except to note that a group of type  $C_4$  is never almost abelian by [GaPT, Cor. 6.2].

Before we go, we note the following:

LEMMA 4.6. *Every 3-dimensional torus over a 3-special field is isotropic.*

*Proof.* Over a 3-special field  $F$ , a torus  $T$  corresponds to a homomorphism  $\phi: \text{Gal}(F) \rightarrow \text{GL}_3(\mathbb{Z})$  that necessarily factors through  $\mathbb{Z}/3\mathbb{Z}$  by Table 4. But any representation  $\mathbb{Z}/3\mathbb{Z} \rightarrow \text{GL}_3(\mathbb{Q})$  admits the trivial representation as direct summand, hence  $T$  is isotropic.  $\square$

## 5. Global fields

We assume in this section that  $F$  is a global field  $F$ . This case is essentially known since the construction of proper subgroups is one of the key ingredients in the proof of Hasse principle, particularly for exceptional types (Harder's method, e.g. [PIR, Ch. VI] and [H]). We prove:

PROPOSITION 5.1. *If an absolutely almost simple group  $G$  over a global field  $F$  is almost abelian, then  $G$  has type  $A_{p-1}$  for some prime  $p$ .*

Conversely, there exist almost abelian groups of type  ${}^1A_{p-1}$  and  ${}^2A_{p-1}$  over  $F$  with  $p$  prime by Proposition 4.1 and Corollary 4.2.

The proof is based on the following consequence of the Hasse Principle:

*If  $F$  is a global field with no real embeddings, then the anisotropic semisimple  $F$ -groups are of type  $A \times \cdots \times A$ .* (5.2)

For the number field case, see [PIR, Th. 6.25]. For the function field case, this is a result of Harder [H, Satz C, p. 133].

The only tricky case of our proof is that of outer type  $A$ . From Prasad-Rapinchuk's study of subalgebras of algebras with involutions [PrR 86, App. A], [PrR 08], we can derive the following fact.

PROPOSITION 5.3. *Let  $L$  be a quadratic field extension of  $F$ . Let  $A$  be a central simple  $L$ -algebra of degree  $n$  equipped with an involution  $\tau$  of the second kind such that  $L^\tau = F$ . If  $\sqrt{-1} \in F$  or if  $n$  is odd, then there exists a cyclic field extension  $E/F$  of degree  $n$  such that  $E \otimes_F L$  is a field and such that  $(E \otimes_F L, \text{Id} \otimes \tau)$  embeds in  $(A, \tau)$ .*

*Proof.* Let  $S$  be a finite set of places such that the group  $\text{SU}(A, \tau)$  is quasi-split for places away from  $S$ . We assume that  $S$  contains a finite place  $v_0$  such that  $L \otimes_F F_v$  is not a field; such a place exists by Chebotarev density and if none are in  $S$ , then we add one.

For each place  $v \in S$ , we pick a cyclic étale  $F_v$ -algebra  $E_v$  of degree  $n$  such that  $(E_v, \text{Id}) \otimes_{F_v} (L \otimes_F F_v, \tau)$  embeds in  $(A, \tau) \otimes_F F_v$  as an  $(L \otimes_F F_v)$ -algebra with involution.

Case 1: Suppose first that  $L \otimes_F F_v$  is not a field. Then  $A \otimes_L (L \otimes_F F_v) \cong A_v \times A_v$  where  $A_v$  is a central  $F_v$ -algebra.

If  $v$  is finite, then by local class field theory,  $A_v$  is split by the unramified extension of  $F_v$  of degree  $n$ . We take it to be  $E_v$ .

If  $v$  is infinite, then  $F_v = \mathbb{C}$  or  $n$  is odd. In either case,  $A \otimes_L (L \otimes_F F_v)$  is split. Hence  $\mathrm{SU}(A, \tau) \times_F F_v$  is split; in this case we delete  $v$  from  $S$ .

Case 2: Suppose now that  $L \otimes_F F_v$  is a field  $L_w$ . We claim that the algebra  $A \otimes_L L_w$  is split. It is obvious in the archimedean case; in the finite case, it follows from the bijectivity of the norm map  $\mathrm{Br}(L_w) \rightarrow \mathrm{Br}(F_v)$ . Then  $(A, \tau) \otimes_F F_v$  is adjoint to a hermitian form for  $L_w/F_v$ . By diagonalizing it, it follows that  $L_w^n$  embeds in  $(A \otimes_F F_v, \tau)$ . We take  $E_v$  to be  $F_v^n$ .

The map

$$H^1(F, \mathbb{Z}/n\mathbb{Z}) \rightarrow \prod_{v \in S} H^1(F_v, \mathbb{Z}/n\mathbb{Z})$$

is onto by [NSW, Cor. 9.2.3] (see also Ex. 1 on p. 459 of that book). So there exists a cyclic étale  $F$ -algebra  $E$  such that  $E \otimes_F F_v$  is isomorphic to  $E_v$  for all  $v \in S$ . Note that  $E$  is a field because  $E_{v_0}$  is. Further,  $L \otimes_F F_{v_0}$  is not a field, but  $E \otimes_F F_{v_0}$  is, so  $E$  does not contain a copy of  $L$ . As  $E$  is Galois over  $F$ , it follows that  $E \otimes L$  is a field.

By construction, for  $v \in S$ , the algebra with involution  $(E_v, \mathrm{Id}) \otimes_{F_v} (L \otimes F_v, \tau)$  embeds in  $(A, \tau) \otimes_F F_v$ . But the same is true for  $v$  not in  $S$ , because the group  $\mathrm{SU}(A, \tau) \times F_v$  is quasi-split, cf. [PIR, p. 340]. By [PrR08, Th. 4.1] (using that  $E \otimes L$  is a field),  $(E, \mathrm{Id}) \otimes (L, \tau)$  embeds in  $(A, \tau)$  as  $F$ -algebras with involution.  $\square$

*Proof of Proposition 5.1.* Let  $G$  be an absolutely almost simple group over  $F$ , not of type  $A_{n-1}$  for  $n$  prime. We prove case-by-case that  $G$  is not almost abelian.

Type  ${}^1A$ : If  $G$  has type  ${}^1A_{n-1}$ , then it is isogenous to  $\mathrm{PGL}_1(A)$  where  $A$  is a central simple algebra of degree  $n$ . By the Brauer-Hasse-Noether theorem,  $A$  is a cyclic algebra, so contains a central simple proper subalgebra  $A'$  if  $n$  is not prime. Then  $\mathrm{PGL}_1(A')$  is a proper semisimple subgroup of  $\mathrm{PGL}_1(A)$ , hence  $\mathrm{PGL}_1(A)$  is not almost abelian.

Type  ${}^2A$ : If  $G$  has type  ${}^2A_{n-1}$  with  $n$  not prime, it is isogenous to  $\mathrm{SU}(B, \tau)$  where  $B$  is a central simple algebra of degree  $n \geq 3$  defined over a quadratic separable extension  $L/F$  and equipped with an involution of the second kind. If  $n$  is even, we know that  $B$  contains a quadratic subalgebra  $L'/L$  stable under  $\tau$  [K, §5.7, p. 109]. So  $\mathrm{SU}(L', \tau)$  is a rank one  $k$ -torus of  $\mathrm{SU}(B, \tau)$ , hence  $\mathrm{SU}(B, \tau)$  is not almost abelian. If  $n$  is odd, Proposition 5.3 shows that  $(B, \tau)$  admits a subalgebra  $(E \otimes_F L, \mathrm{Id} \otimes \tau)$  where  $E$  is a cyclic Galois  $F$ -algebra of degree  $n$ . Because  $n$  is not prime, there is a field  $E_0$  such that  $F \subsetneq E_0 \subsetneq E$ , and the special unitary group of  $(E_0 \otimes L, \mathrm{Id} \otimes \tau)$  is a non-maximal, non-zero torus in  $\mathrm{SU}(B, \tau)$ . Again  $\mathrm{SU}(B, \tau)$  is not almost abelian.

Other types: By (5.2), we can assume that  $F$  has real places. Then we write  $P$  for the parabolic subgroup of  $G$  defined over  $F(\sqrt{-1})$  from Theorem 5.2. It follows that  $G$  has an  $F$ -subgroup  $L$  that is conjugate over  $F(\sqrt{-1})$  to a Levi subgroup of  $P$  [PIR, p. 383, Lemma 6.17'], hence  $G$  is not almost abelian.  $\square$

## 6. Versal groups

We first recall the definition of a versal torsor.

**DEFINITION 6.1.** (Serre, [Se03, §5]) Let  $G$  be a smooth affine algebraic group defined over a field  $F_0$ . A *versal  $G$ -torsor*  $P$  is a  $G$ -torsor defined over a finitely generated extension  $F$  of  $F_0$  such that there exists a smooth, irreducible variety  $X$  over  $F_0$  with function field  $F$  and a  $G$ -torsor  $\mathfrak{Q} \rightarrow X$ , with base  $X$ , with the following two properties:

- (1) The fiber of  $\mathfrak{Q}$  at the generic point of  $X$  is  $P$ .

(2) For every extension  $E$  of  $F_0$ , with  $E$  infinite, every  $G$ -torsor  $T$  over  $E$ , and every nonempty open subvariety  $U$  of  $X$ , there exists  $x \in U(E)$  such that the fiber  $\Omega_x$  is isomorphic to  $T$ .

Versal torsors exist and one construction is to take a faithful linear representation  $G \rightarrow \mathrm{GL}_N$  and the torsor  $\mathrm{GL}_N \rightarrow X = \mathrm{GL}_N/G$ . Here we mimic this definition for versal groups, which are the “most general” groups of a given quasi-split type.

**DEFINITION 6.2.** Fix a field  $F_0$ , a connected Dynkin diagram  $\Delta$ , and a class  $\phi \in H^1(F_0, \mathrm{Aut}(\Delta))$ , namely a continuous homomorphism of the absolute Galois group  $\mathrm{Gal}(F_0)$  by graph automorphisms on  $\Delta$ , up to conjugation by an element of  $\mathrm{Aut}(\Delta)$ . A versal group of type  $(\Delta, \phi)$  is an adjoint semisimple group  $G$  defined over a finitely generated extension  $F$  of  $F_0$  and of quasi-split type  $(\Delta, \phi_F)$  such that there exists a smooth, irreducible variety  $X$  over  $F_0$  with function field  $F$  and an adjoint semisimple group scheme  $\mathfrak{G}/X$  with the following two properties:

- (i) The fiber of  $\mathfrak{G}$  at the generic point of  $X$  is  $G$ .
- (ii) For every extension  $E$  of  $F_0$ , with  $E$  infinite, every adjoint semisimple group  $H/E$  of quasi split type  $(\Delta, \phi_E)$  and every nonempty open subvariety  $U$  of  $X$ , there exists  $x \in U(E)$  such that the fiber  $\mathfrak{G}_x$  is isomorphic to  $H$ .

Versal groups exist. There is a unique quasi-split simple adjoint group  $G^q$  of quasi-split type  $(\Delta, \phi)$ . We embed  $G^q$  in some  $\mathrm{GL}_N$ , put  $X = \mathrm{GL}_N/G^q$  and  $F = F_0(X)$ . By étale descent [Mi 80, p. 134], we can twist by inner automorphisms  $G^q \times_{F_0} X$  by the  $G^q$ -torsor  $\mathrm{GL}_N \rightarrow X$ , this defines  $\mathfrak{G}/X$  and its generic class  $G/F$ . Since every adjoint semisimple group  $H/E$  of quasi-split type  $(\Delta, \phi_E)$  is obtained by twisting by a  $G^q$ -torsor over  $E$ , it follows that  $G/F$  is a versal group of quasi-split type

A *versal group* is a group  $G$  (over  $F$ ) obtained in this manner from some  $F_0, \Delta, \phi$ . As an abbreviation, we say that  $G$  is a *versal group of type*  ${}^aT_n$ , for  $T_n$  the Killing-Cartan type of  $\Delta$  and  $a = |\mathrm{im} \phi|$ .

There is a further notion of versal groups when we allow the  $*$ -action to vary inside  $H^1(E, \mathrm{Aut}(\Delta))$  with image in the conjugate in a subgroup  $\Gamma \subset \mathrm{Aut}(\Delta)$ .

**DEFINITION 6.3.** Fix a field  $F_0$ , a connected Dynkin diagram  $\Delta$ , and a subgroup  $\Gamma \subset \mathrm{Aut}(\Delta)$ . A *superversal group* of type  $(\Delta, \Gamma)$  is an adjoint semisimple group  $G$  defined over a finitely generated extension  $F$  of  $F_0$  and of quasi-split type  $(\Delta, \phi_G)$  such that

$$\phi_G \in \mathrm{Im}\left(H^1(F, \Gamma) \rightarrow H^1(F, \mathrm{Aut}(\Delta))\right)$$

and such that there exists a smooth, irreducible variety  $X$  over  $F_0$  with function field  $F$  and an adjoint semisimple group scheme  $\mathfrak{G}/X$  with the following two properties:

- (i) The fiber of  $\mathfrak{G}$  at the generic point of  $X$  is  $G$ .
- (ii) For every extension  $E$  of  $F_0$ , with  $E$  infinite, every adjoint semisimple group  $H/E$  of quasi-split type  $(\Delta, \phi_H)$  such that

$$\phi_H \in \mathrm{Im}\left(H^1(E, \Gamma) \rightarrow H^1(E, \mathrm{Aut}(\Delta))\right) \tag{6.4}$$

and every nonempty open subvariety  $U$  of  $X$ , there exists  $x \in U(E)$  such that the fiber  $\mathfrak{G}_x$  is isomorphic to  $H$ .

Similarly, superversal groups exist. Let  $G^d$  be the split adjoint split group of type  $\Delta$ . Fix a subgroup  $\Gamma \subset \text{Aut}(\Delta)$ . Write  $\text{Aut}_\Gamma(G^d)$  for the pull back by the inclusion  $\Gamma \subset \text{Aut}(\Delta)$  of the exact sequence

$$1 \rightarrow G^d \rightarrow \text{Aut}(G^d) \rightarrow \text{Aut}(\Delta) \rightarrow 1.$$

We embed  $\text{Aut}_\Gamma(G^d)$  in some  $\text{GL}_N$ , put  $X = \text{GL}_N / \text{Aut}_\Gamma(G^d)$  and  $F = F_0(X)$ . We twist by  $G^a \times_{F_0} X$  by the  $\text{Aut}_\Gamma(G^a)$ -torsor  $\text{GL}_N \rightarrow X$ , this defines  $\mathfrak{G}/X$  and its generic class  $G/F$ . We shall see below (6.5) that every adjoint semisimple group  $H/E$  of quasi-split type  $(\Delta, \phi_H)$  satisfying the condition (6.4) is obtained by twisting  $G^a$  by a  $\text{Aut}_\Gamma(G^a)$ -torsor over  $E$ , it follows that  $G/F$  is a superversal group of type  $(\Delta, \Gamma)$

A *superversal group* is a group  $G$  (over  $F$ ) obtained in this manner from some  $F_0, \Delta, \Gamma$ . As an abbreviation, we say that  $G$  is a *superversal group of type*  ${}^aT_n$ , for  $T_n$  the Killing-Cartan type of  $\Delta$  and  $a = |\Gamma|$ .

The case  $\Gamma = 1$ , is nothing but the versal inner form.

6.5. Maintain the notation of Definition 6.3. Suppose that  $M$  is an adjoint group with the same Dynkin diagram  $\Delta$ , defined over an extension  $E$  of  $F_0$ , and the  $*$ -action on the Dynkin diagram of  $M$  belongs to the image of  $H^1(E, \Gamma) \rightarrow H^1(E, \text{Aut}(\Delta))$ . We claim that  $M$  is in the image of the map  $H^1(E, \text{Aut}_\Gamma(G^d)) \rightarrow H^1(E, \text{Aut}(G^d))$ .

We have to check that this kind of groups occur as twisted forms of  $G^d$  by cocycles with value in  $\text{Aut}_\Gamma(G^d)$ . There exists  $[z] \in H^1(E, \text{Aut}(G^d))$  such that  $M \cong G_z^d$ . We consider the commutative diagram

$$\begin{array}{ccc} H^1(E, \text{Aut}(G^d)) & \xrightarrow{\pi} & H^1(E, \text{Aut}(\Delta)) \\ \uparrow & & \uparrow \\ H^1(E, \text{Aut}_\Gamma(G^d)) & \longrightarrow & H^1(E, \Gamma). \end{array}$$

Our hypothesis is that  $\pi_*[z] \in \text{Im}\left(H^1(E, \Gamma) \rightarrow H^1(E, \text{Aut}(\Delta))\right)$ . According to [Se 02, §I.5, prop. 37], this is equivalent to the fact that  $\pi_{*(z)}(\text{Aut}(\Delta)/\Gamma)(E) \neq \emptyset$ . This finite set is nothing but  $\pi_{*(z)}(\text{Aut}(G^d)/\text{Aut}_\Gamma(G^d))(E)$  hence the claim.

We conclude this section by the following observation.

LEMMA 6.6. *Let  $\Psi$  be a root datum in the sense of Springer [Sp, 7.4.1]. The following are equivalent:*

- (i) *The superversal group  $G/F$  contains a reductive subgroup of type  $\Psi$ .*
- (ii) *For every field extension  $E/F_0$  and every adjoint group  $M/E$  of type  $\Delta$  such that  $\varphi_M \in \text{Im}\left(H^1(E, \Gamma) \rightarrow H^1(E, \text{Aut}(\Delta))\right)$ , the group  $M/E$  has a reductive subgroup of type  $\Psi$ .*

Note that there is an analogous statement for versal groups.

We use a root datum instead of a root system so as to include also tori and the kernel of the map from the simply connected cover.

*Proof.* The statement does not depend of the choice of superversal group. We can then work with our construction, i.e., with an  $\text{Aut}_\Gamma(G)$ -torsor  $V/F$  arising as the generic fiber of a  $\text{Aut}_\Gamma(G)$ -torsor  $\mathfrak{V}/X$  where  $X$  is a smooth  $F_0$ -variety such that  $F = F_0(X)$  and  $\mathfrak{G}/X$  the twisted  $X$ -form of  $G^d$  by  $\mathfrak{V}$ . The implication (2)  $\implies$  (1) is obvious. Conversely, assume that  $G/F$  admits a reductive  $F$ -subgroup  $H \subset G/F$  of type  $\Psi$ . The idea is to extend  $H$  locally. The quotient  $Q = G/H$  is a  $G$ -variety over  $F$ . According to [SGA3, VI<sub>B</sub>.10.16] there exists an open (non empty) Zariski-connected subset  $U \subset X$  and a homogeneous  $\mathfrak{G}$ -scheme  $\Omega/U$  such that

$Q = \Omega \times_U F_0(X)$ . Shrinking  $U$  further if necessary, we can assume that the origin of  $Q$  extends to a point  $q \in \Omega(U)$ . We define the closed  $U$ -subgroup scheme  $\mathfrak{H} := \text{Stab}_{\mathfrak{G}}(q)$ . Again, shrinking  $U$  if necessary, we can assume that  $\mathfrak{H}/U$  is smooth and connected. By [SGA3, XIX.2.6],  $\mathfrak{H}/U$  is a reductive group scheme. By Demazure’s type unicity theorem [SGA3, XXII.2.8],  $\mathfrak{H}$  is of type  $\Psi$ , i.e., all geometric fibers are of type  $\Psi$ .

We can now proceed to the proof. We are given a field  $E/F_0$  and an adjoint group  $M/E$  of type  $\Delta$  such that its  $*$ -action  $\varphi_M \in H^1(E, \text{Aut}(\Delta))$  comes from  $H^1(F, \Gamma)$ . By 6.5, there exists  $x \in U(E)$  such that  $M$  is isomorphic to the fiber  $x^*\mathfrak{G}$ . Then  $M/E$  has the closed subgroup  $x^*\mathfrak{H}$  which is reductive of type  $\Psi$ .  $\square$

**COROLLARY 6.7.** *Let  $G$  be a superserial group as in Definition 6.3. The following are equivalent:*

- (1) *The superserial group  $G/F$  is not almost abelian.*
- (2) *For every field extension  $E/F_0$  and every adjoint group  $M/E$  of type  $\Delta$  such that  $\varphi_M \in \text{Im}(H^1(E, \Gamma) \rightarrow H^1(E, \text{Aut}(\Delta)))$ , the group  $M/E$  is not almost abelian.*

*Proof.* Again (2)  $\implies$  (1) is obvious. Conversely, assume that  $G/F$  admits a proper non zero  $F$ -subgroup  $H \subset G$ . Since  $G$  is anisotropic,  $H$  is reductive. Hence Lemma 6.6 applies.  $\square$

### 7. Rost invariant and adjoint groups

7.1. Let  $G^q$  be a quasi-split simple adjoint algebraic group, and write  $\tilde{G}^q$  for its simply connected cover. The Rost invariant of  $\tilde{G}^q$  is a morphism of functors  $\text{Fields}/F \rightarrow \text{Sets}$ :

$$r_{\tilde{G}^q} : H^1(*, \tilde{G}^q) \rightarrow H^3(*, \mathbb{Z}/\delta\mathbb{Z}(2)),$$

where  $\delta$  is the Dynkin index of  $\tilde{G}^q$  as defined in [Mer 03, §10]. (We remind the reader that in case the characteristic of  $F$  divides  $\delta$ , the group  $H^3(F, \mathbb{Z}/\delta\mathbb{Z}(2))$  is defined using logarithmic differential forms as in [Ka], see [Mer 03, App. A]. In case the characteristic of  $F$  does not divide  $d$ , it is the Galois cohomology group  $H^3(F, \mu_d^{\otimes 2})$ .) Factor  $\delta$  as  $\delta_0 m$ , where  $m$  is maximal with the property of being relatively prime to both  $\delta_0$  and the exponent of the group  $Z^*(F_{\text{sep}})$  of  $F_{\text{sep}}$ -points of the cocenter of  $\tilde{G}^q$ . The number  $m$  is 1 exactly for  $G^q$  of type  ${}^1A_n, B_n$ , or  $C_n$  for all  $n$ ; type  ${}^2A_n$  for  $n$  odd; or type  ${}^1D_n$  or  ${}^2D_n$  for  $n \geq 4$ . The number  $m$  is  $\delta$  when  $Z$  is trivial, i.e., for  $G^q$  of type  $E_8, F_4$ , or  $G_2$ . All the other cases—i.e., where  $m \neq 1, \delta$ —are given in Table 7.

type of $G^q$	$m$	Dynkin index $\delta$ of $\tilde{G}^q$	exponent of $Z^*(F_{\text{sep}})$
${}^2A_n$ ( $n$ even)	2	2	$n + 1$
${}^3D_4, {}^6D_4$	3	6	2
${}^1E_6$	2	6	3
${}^2E_6$	4	12	3
$E_7$	3	12	2

TABLE 7. Factorizations of Dynkin indexes for some quasi-split absolutely simple groups. All other quasi-split absolutely simple groups have  $m = 1$  or  $m = \delta$ .

For each extension  $K/F$ , the group  $H^3(K, \mathbb{Z}/\delta\mathbb{Z}(2))$  can naturally be written as a direct sum of its  $m$  and  $\delta_0$  torsion parts, i.e., as  $H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \oplus H^3(K, \mathbb{Z}/\delta_0\mathbb{Z}(2))$ .

PROPOSITION 7.2. *Suppose the characteristic of  $F$  does not divide  $m$ . Then there is a unique invariant*

$$r_{G^q, m} : H^1(*, G^q) \rightarrow H^3(*, \mathbb{Z}/m\mathbb{Z}(2))$$

such that the diagram

$$\begin{array}{ccc} H^1(*, \tilde{G}^q) & \xrightarrow{r_{\tilde{G}^q}} & H^3(*, \mathbb{Z}/\delta\mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ H^1(*, G^q) & \xrightarrow{r_{G^q, m}} & H^3(*, \mathbb{Z}/m\mathbb{Z}(2)) \end{array}$$

commutes.

*Proof.* The result is trivial if  $\tilde{G}^q = G^q$  (i.e.,  $m = \delta$ ) or  $m = 1$ , so we may assume that  $m$  is given by Table 7. In particular,  $m$  is a power of some prime  $p$ .

Fix an element  $y \in H^1(K, G)$  for some extension  $K/F$ ; we claim that there is an extension  $L/K$  such that  $\text{res}_{L/K}(y)$  is in the image of  $H^1(L, \tilde{G}^q) \rightarrow H^1(L, G)$  and  $[L : K]$  is relatively prime to  $m$ . Let  $K_p$  denote the co- $p$ -closure of the perfect closure of  $K$ . The group  $H_{\text{fppf}}^2(K_p, Z)$  is zero because the exponent of  $Z$  is not divisible by  $p$ . The exactness of the sequence  $1 \rightarrow Z \rightarrow \tilde{G}^q \rightarrow G^q \rightarrow 1$  implies that there is a finite extension  $E/K$  and an  $x \in H^1(E, \tilde{G}^q)$  that maps to  $\text{res}_{E/K}(y) \in H^1(E, G^q)$ . We take  $L$  to be the separable closure of  $K$  in  $E$ ; the sets  $H^1(L, \tilde{G}^q)$  and  $H^1(E, \tilde{G}^q)$  are identified by the restriction map, and the claim is proved.

Now suppose that  $x_1, x_2 \in H^1(L, \tilde{G}^q)$  both map to  $y$ . Then there is a  $z \in H_{\text{fppf}}^1(L, Z)$  such that  $z \cdot x_1 = x_2$ . Applying the Rost invariant, we find

$$r_{\tilde{G}^q}(x_1) = r_{\tilde{G}^q}(z \cdot x_2) = r_{\tilde{G}^q}(x_2)$$

by [Gi, p. 76, Lemma 7], cf. [GaQ, Remark 2.5(i)]. Combining this with the previous paragraph proves the existence and uniqueness of  $r_{G^q, m}$  by [Ga09, Proposition 7.1] (which uses the hypothesis on the characteristic of  $F$ ).  $\square$

DEFINITION 7.3. Let  $G$  be a simple algebraic group over  $F$ , and write  $\overline{G}$  for its associated adjoint group. There is a unique class  $\eta \in H^1(F, \overline{G})$  such that the group  $\overline{G}_\eta$  obtained by twisting  $\overline{G}$  by  $\eta$  is quasi-split [KMRT, 31.6]. Put  $m$  for the natural number defined for the quasi-split group  $\overline{G}_\eta$  in 7.1, and suppose that the following holds:

$$\text{The characteristic of } F \text{ does not divide } m \text{ or } G \text{ has type } F_4, G_2, \text{ or } E_8. \tag{7.4}$$

Write  $\theta_\eta$  for the twisting isomorphism  $H^1(F, \overline{G}_\eta) \xrightarrow{\sim} H^1(F, \overline{G})$  and define:

$$r(G) := r_{\overline{G}_\eta, m}(\theta_\eta^{-1}(0)) \in H^3(F, \mathbb{Z}/m\mathbb{Z}(2)) \subseteq H^3(F, \mathbb{Q}/\mathbb{Z}(2)).$$

This element depends only on the isomorphism class of  $G$ . If  $G$  is one of the groups listed in Table 7, then  $m$  divides 24 and  $r(G)$  takes values in  $H^3(F, \mathbb{Z}/m\mathbb{Z})$  [KMRT, §VII, exercise 11].

It is easy to see that  $r$  is an invariant in the sense of [Se03], e.g., for every extension  $K/F$  we have:

$$\text{res}_{K/F}(r(G)) = r(G \times_F K).$$

LEMMA 7.5. *If  $G$  is a versal group and (7.4) holds, then  $r(G)$  has order  $m$ .*

*Proof.* The claim is equivalent to the following statement in the language of Proposition 7.2: If  $V$  is a versal  $G^q$ -torsor, then  $r_{G^q, m}(V)$  has order  $m$  in  $H^3(F, \mathbb{Z}/m\mathbb{Z}(2))$ . To prove this, by the specialization property of versal torsors [Se03, 12.3], it suffices to produce an element

$y \in H^1(E, G^q)$  for some extension  $E/F$  such that  $r_{G^q, m}(y)$  has order  $m$  in  $H^3(E, \mathbb{Z}/m\mathbb{Z}(2))$ . We take  $y$  to be the image of a versal  $\tilde{G}^q$ -torsor  $\tilde{V}$ . By [Mer 03, 10.8], the element  $r_{\tilde{G}^q}(\tilde{V})$  has order  $\delta$ , and the claim follows.  $\square$

## 8. Tits algebras

Let  $G$  be a semisimple algebraic group over  $F$ . Tits defined certain invariants  $\beta_G(\chi)$  of  $G$  in [Ti 71] with the following properties, cf. [KMRT, §27], [MPW1, §2], and [Mer 96, §4]. Write  $Z$  for the scheme-theoretic center of a simply connected cover of  $G$ . The Cartier dual  $Z^*$  of the center—the *cocenter*—is an étale group scheme. For  $\chi \in Z^*(F_{\text{sep}})$ , we put  $F_\chi$  for the subfield of  $F_{\text{sep}}$  fixed by the stabilizer of  $\chi$  in  $\text{Gal}(F)$ . The *Tits algebra*  $\beta_G(\chi)$  is an element of  $H^2(F_\chi, \mathbb{G}_m)$ , the Brauer group of  $F_\chi$ .

Let  $t_G \in H^2_{\text{fppf}}(F, Z)$  be the Tits class of  $G$  [KMRT, §31]. We note the following properties:

(8.1) For all  $\chi \in Z^*(F_{\text{sep}})$ , the image of  $t_G$  under the composite map

$$H^2_{\text{fppf}}(F, Z) \xrightarrow{\text{res}} H^2_{\text{fppf}}(F_\chi, Z) \xrightarrow{\chi^*} H^2(F_\chi, \mathbb{G}_m)$$

is  $\beta_G(\chi)$ , see [KMRT, 31.7].

(8.2) For  $G$  of inner type, the Galois group acts trivially on  $Z^*(F_{\text{sep}})$  and  $F_\chi = F$  for all  $\chi$ .

(8.3) The Galois action on  $Z^*$  is “compatible” with the Tits algebras in the following sense: For  $\sigma \in \text{Gal}(F)$ , clearly  $F_{\sigma\chi}$  equals  $\sigma(F_\chi)$ . Further, the natural map

$$\sigma: H^2(F_\chi, \mathbb{G}_m) \rightarrow H^2(F_{\sigma\chi}, \mathbb{G}_m)$$

sends  $\beta_G(\chi)$  to  $\beta_G(\sigma\chi)$ . This follows from (8.1). In particular, the index of the division algebra representing  $\beta_G(\chi)$  is the same as the index of the algebra representing  $\beta_G(\sigma\chi)$ .

(8.4) The Tits algebras are compatible with scalar extension in a natural way. If  $K/F$  is an extension contained in  $F_{\text{sep}}$ , then  $K_\chi$  is the compositum of  $F_\chi$  and  $K$ , and  $\beta_{G \times K}(\chi)$  is the restriction  $\text{res}_{K_\chi/F_\chi} \beta_G(\chi)$ .

(8.5) For every integer  $k$ , the element  $\text{res}_{F_\chi/F_{k\chi}} \beta_G(k\chi)$  equals  $k \cdot \beta_G(\chi)$  in  $H^2(F_\chi, \mathbb{G}_m)$ , by (8.1).

We say simply the *index of  $\beta_G(\chi)$*  to mean the index of a division algebra representing  $\beta_G(\chi)$ , equivalently, the dimension of the smallest separable extension of  $F_\chi$  that kills  $\beta_G(\chi)$  [GiS, 4.5.9]—or, what is the same, the subgroup of  $H^2(F_\chi, \mathbb{G}_m)$  generated by  $\beta_G(\chi)$ . Property (8.5) gives:

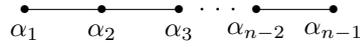
(8.6) *If  $k$  is relatively prime to the exponent of  $Z^*(F_{\text{sep}})$ , then  $\beta_G(k\chi)$  and  $\beta_G(\chi)$  have the same index.* Indeed,  $\chi$  and  $k\chi$  generate the same subgroup of  $Z^*(F_{\text{sep}})$ , hence  $F_{k\chi}$  equals  $F_\chi$ .

Define  $n_G(\chi)$  to be  $\text{gcd}\{\dim \rho\}$ , where  $\rho$  varies over the representations of  $G$  defined over  $F_{\text{sep}}$  such that  $\rho|_Z = \chi$ . For example,  $n_G(0) = 1$ . Note that  $n_G(\chi)$  depends only on  $\chi$  and the Killing-Cartan type of  $G$ , and not on the field  $F$ . Further, a twisting argument together with (8.1) shows that the index of  $\beta_G(\chi)$  divides  $n_G(\chi)$ , see [MPW1, Prop. 2.4]. The numbers  $n_G(\chi)$  have been computed in [MPW1] and [MPW2].

**EXAMPLE 8.7.** In case  $Z^*(F_{\text{sep}})$  is  $\mathbb{Z}/p\mathbb{Z}$  for some prime  $p$ , [MPW2, Prop. 6.13(2)] gives a convenient formula. One finds for nonzero  $\chi$ :

$$n_G(\chi) = \begin{cases} 27 & \text{for } G \text{ of type } {}^1E_6 \text{ or } {}^2E_6 \text{ by [MPW2, p. 156]} \\ 8 & \text{for } G \text{ of type } E_7 \text{ by [MPW2, p. 166].} \end{cases}$$

EXAMPLE 8.8. Label the simple roots of a group of type  $A_{n-1}$  as in the Dynkin diagram:



Write  $\omega_i$  for the fundamental dominant weight corresponding to the simple root  $\alpha_i$ . Pages 562–564 in [MPW1] verify from the definition of  $n_G$  that  $n_G(\omega_i|_Z) = n/\gcd(i, n)$ .

Regarding the Tits algebras of versal/superversal groups, our main tool is the following theorem of Merkurjev:

THEOREM 8.9 (Th. 5.2 of [Mer 96]). *If  $G$  is a versal group and  $\chi$  is fixed by  $\text{Gal}(F)$ , then the index of the Tits algebra  $\beta_G(\chi)$  is  $n_G(\chi)$ .*

Alternatively, one can find concrete constructions for  $G$  of inner type in [MPW1] and [MPW2]. We extend this theorem in §14 below.

### 9. Groups of type $E_8$

LEMMA 9.1. *If  $G$  is a group of type  $E_8$  that is neither split nor anisotropic, then the semisimple anisotropic kernel of  $G$  and the order of  $r(G)$  are as in one of the rows of Table 9.*

semisimple anisotropic kernel of $G$	order of $r(G)$
$E_7$	$\neq 1$ , divides 12
$D_7$	2
$E_6$	3 or 6
$D_6$	2
$D_4$	2

TABLE 9. Possible invariants of a group  $G$  of type  $E_8$  that is neither split nor anisotropic.

*Proof (sketch).* The list of possibilities for the semisimple anisotropic kernel  $G^{\text{an}}$  is from [Ti 66] or [Sp]. It remains to verify the claim on  $r(G)$  in each case. As  $G$  is simply connected with trivial Tits algebras and the  $*$ -action is trivial, it follows that the same holds for  $G^{\text{an}}$ . (To see this for the Tits algebras, one applies [Ti 71, p. 211].)

By Tits’s Witt-type theorem from [Ti 66, 2.7.1] or [Sp, 16.4.2], there is a class  $z \in H^1(F, G^{\text{an}})$  such that twisting  $G^{\text{an}}$  and  $G$  by  $z$  gives split groups. The definition of  $r(G)$  says that it equals  $r_{G_z^{\text{an}}}(\theta_z^{-1}(0))$ , so its order divides the Dynkin index of  $G^{\text{an}}$ . By hypothesis,  $G^{\text{an}}$  is anisotropic, hence  $z$  is not zero and the main result of [Ga 01] gives that  $r(G)$  is not zero. The remaining possibilities for the order of  $r(G)$  are those listed in Table 7.  $\square$

LEMMA 9.2 (Cf. [Ti 91, p. 134, Proposition 5]). *Let  $G$  be a group of type  $E_8$  over a 5-special field  $F$ . If  $G$  is not split, then  $G$  is anisotropic and every proper, nontrivial reductive subgroup of  $G$  is a torus of rank 4 or 8 or has semisimple type  $A_4$  or  $A_4 \times A_4$ .*

We note that all of these possibilities can occur, see [Ga 09, 15.7].

*Proof.* As  $G$  is not split, if it is isotropic, then  $r(G)$  is not zero by Table 9. But  $G$  is split by a separable extension of  $F$ , so  $r(G)$  has order dividing 5. It follows from Table 9 that  $G$  is anisotropic. This implies by Table 4 that  $G$  cannot contain tori of ranks 1, 2, or 3. By the previous sentence, every nontrivial reductive subgroup of  $G$  has rank 4 or 8. We claim that any semisimple group  $H$  of rank 4 over  $F$  is isotropic or of type  $A_4$ . Indeed, since  $F$  has no separable field extension of degree  $\leq 4$ , the  $*$ -action of the absolute Dynkin diagram  $\Delta_H$  of  $H$  is trivial, so  $H$  is an inner form. If  $\Delta_H$  contains a summand of type  $A_n$  for  $1 \leq n \leq 3$ ;  $B_n$  for  $2 \leq n \leq 4$ ;  $C_3$ ;  $C_4$ ; or  $D_4$ , then  $H$  is isotropic. If  $H$  is of type  $F_4$ , it contains a subgroup of type  $D_4$ , hence splits as well. So the only possibility for non isotropic  $H$  is to be of type  $A_4$ .

For a subgroup of rank 8, we consult the list of such in [Dy, Table 10]; in addition to  $A_4 \times A_4$ , we find:

- (i)  $A_7 \times A_1, A_5 \times A_2 \times A_1, E_6 \times A_2, E_7 \times A_1, D_6 \times A_1^{\times 2}, D_5 \times A_3, A_3^{\times 2} \times A_1^{\times 2}$
- (ii)  $A_8, D_8, A_2^{\times 4}, D_4^{\times 2}, D_4 \times A_1^{\times 4}, A_1^{\times 8}$

Those in list (1) do not occur because they contain tori of ranks  $\neq 4, 8$ . Those in list (2) do not occur because they are isotropic over a 5-special field.  $\square$

LEMMA 9.3. *Let  $G$  be a non-split group of type  $E_8$  over a 3-special field.*

- (1) *If  $G$  is isotropic, it has rank 2 and semisimple anisotropic kernel of type  $E_6$ .*
- (2)  *$G$  does not contain a semisimple subgroup of type  $A_4$  or  $A_4 \times A_4$ .*

*Proof.* Over a 3-special field, every group of type  $E_7$  is isotropic [Ga 09, 13.1], so (1) follows from Table 9. Since the field is 3-special, a semisimple group of type  $A_4$  or  $A_4 \times A_4$  is split, contradicting the statement that  $G$  has rank at most 2.  $\square$

THEOREM 9.4. *A versal group of type  $E_8$  is almost abelian.*

*Proof.* Every group of type  $E_8$  is simply connected, so the versal group  $G$  is obtained by twisting the split group  $G^q$  of type  $E_8$  by a versal torsor. By [Mer 03, 16.8], the Rost invariant  $r_{G^q}(G)$  has order 60 in  $H^3(F, \mathbb{Z}/60\mathbb{Z}(2))$ . In particular, this element is not killed by a co-3-closure nor a co-5-closure of  $F$  and so  $G$  is not split over such fields. Combining Lemmas 9.2 and 9.3, it suffices to show that  $G$  does not contain any rank 4 tori.

For sake of contradiction, suppose that  $G$  has a rank 4 torus  $S$  defined over  $F$ , and that an extension  $L/F$  splits  $S$ . Then  $G$  has  $L$ -rank at least 4, and so  $G$  is  $L$ -split or has semisimple anisotropic kernel a strongly inner form of  $D_4$  over  $L$ ; in either case  $G$  is split by an extension of  $F$  of degree dividing  $2[L : F]$ . By [To], the torsion index of the compact real Lie group  $E_8$  is 2880, hence every extension that splits  $G$  has dimension divisible by 2880, hence  $[L : F]$  is divisible by  $1440 = 2^5 \cdot 3^2 \cdot 5$ . But by Table 4, every rank 4 torus over  $F$  is split by an extension of degree dividing  $2^7 \cdot 3^2$  or  $2^3 \cdot 5$  or  $2^4 \cdot 3 \cdot 5$ . This is a contradiction.  $\square$

Applying 2.4 gives:

COROLLARY 9.5 (Cf. [Ti 92b, p. 1135, Cor. 3]). *Inside a versal group  $G$  of type  $E_8$  over a field of characteristic 0, the proper nontrivial connected subgroups are the maximal tori.*

In the proof of Theorem 9.4, the second paragraph amounts to a reference to Totaro's general result on the torsion index. Alternatively, one could replace it with a reference to [Ti 92b, Lemme 5], which says: *If a group  $G$  of type  $E_8$  contains a regular rank 4 torus, then  $G$  is split*

by an extension of dimension dividing  $2^7 \cdot 3^2$  or  $2^3 \cdot 5$ . This gives a strengthening of Theorem 9.4:

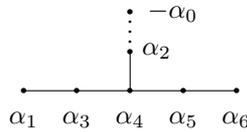
**THEOREM 9.6.** *If  $G$  is a group of type  $E_8$  such that  $r(G)$  has order 60, then  $G$  is almost abelian.*

*Proof.* An extension of degree dividing  $2^7 \cdot 3^2$  or  $2^3 \cdot 5$  cannot kill an element of order 60 in  $H^3(F, \mathbb{Z}/60\mathbb{Z}(2))$  by a restriction/corestriction argument, so  $G$  cannot contain a rank 4 torus. Now combine Lemmas 9.2 and 9.3(2).  $\square$

### 10. Weyl group of $E_6$

This section prepares the ground for the proof of Proposition 11.1.

10.1. We label the simple roots in a root system of type  $E_6$  as in the extended Dynkin diagram



where  $\alpha_0$  denotes the highest root. The roots

$$\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\alpha_0 \tag{10.2}$$

are a basis for a root subsystem of type  $A_2 \times A_2 \times A_2$ , and we refer to it simply as  $A_2^{\times 3}$ . Of course, all subsystems of  $E_6$  of that type are conjugate under the Weyl group  $W(E_6)$  [**Bou**, Exercise VI.4.4].

**LEMMA 10.3.** *There is exactly one conjugacy class of elements of  $W(E_6)$  of order 2, determinant 1, and trace  $-2$ . Every element of the class normalizes a sub-root-system of type  $A_2^{\times 3}$ .*

Here, the determinant and trace are computed relative to the action of  $W(E_6)$  on the  $\mathbb{Q}$ -vector space spanned by the roots.

*Proof of Lemma 10.3.* The determinant defines an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow W(E_6) \xrightarrow{\det} \pm 1 \longrightarrow 1$$

where  $\Gamma$  is the finite simple group  $U_4(2)$ , cf. [**Bou**, Exercise VI.4.2]. To prove the first claim, it suffices to note that there is exactly one conjugacy class of elements of  $\Gamma$  of order 2 and trace  $-2$ , as can be read off of the 4th line of the table on p. 27 of the Atlas [**Atlas**].

For the second claim, take  $w$  to be the composition of the reflections in the (pairwise strongly orthogonal) roots

$$\begin{matrix} \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 & \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\ \alpha_5 + \alpha_6 & \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \end{matrix}$$

Clearly,  $w$  has determinant 1. It interchanges  $\alpha_1, \alpha_3$  with  $\alpha_2, -\alpha_0$  respectively and sends  $\alpha_5, \alpha_6$  to  $-\alpha_5, -\alpha_6$ . It follows that  $w$  has trace  $-2$  and stabilizes the subsystem  $A_2^{\times 3}$ .  $\square$

EXAMPLE 10.4. The roots  $\alpha_2, \alpha_3, \alpha_4, \alpha_5$  span a sub-root-system of type  $D_4$  in  $E_6$ , and this gives an inclusion  $W(D_4) \hookrightarrow W(E_6)$ . The element  $-1$  of  $W(D_4)$  maps to an element  $w \in W(E_6)$  of order 2. This element acts as  $-1$  on the 4-dimensional subspace spanned by the roots in  $D_4$  and fixes the (orthogonal) 2-dimensional subspace spanned by the fundamental weights of  $E_6$  dual to  $\alpha_1$  and  $\alpha_6$ . It follows that  $w$  has determinant 1 and trace  $-2$ .

11. Groups of type  ${}^1E_6$

Let  $G$  be a group of type  ${}^1E_6$ . The cocenter  $Z^*$  is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ . The numbers  $n_G(\chi)$  are 27 for nonzero  $\chi$ , and the Tits algebras  $\beta_G(1)$  and  $\beta_G(2)$  have the same index by (8.6). We prove:

PROPOSITION 11.1. *Let  $G$  be a group of type  ${}^1E_6$  over a field  $F$  of characteristic  $\neq 2$ .*

- (1) *If  $F$  is a  $p$ -special field for some prime  $p$ , then  $G$  contains a semisimple  $F$ -subgroup of type  $A_2^{\times 3}$ .*
- (2) [Ti 91, p. 129] *If  $\beta_G(1)$  has index 27, then  $G$  has no nontrivial, proper reductive  $F$ -subgroups other than maximal tori and possibly semisimple subgroups of type  $A_2^{\times 3}$ .*

Versal groups have  $\beta_G(1)$  of index 27, so (2) is not empty. (Alternatively, an explicit construction of a group of type  ${}^1E_6$  as in (2) and containing a semisimple subgroup of type  $A_2^{\times 3}$  can be found in [MPW2, pp. 153–155].)

LEMMA 11.2. *If a group  $G$  of type  ${}^1E_6$  over a field of characteristic  $\neq 2$  is neither split nor anisotropic, then its semisimple anisotropic kernel, the index of  $\beta_G(1)$ , and  $r(G)$  are as in one of the rows of Table 11.*

semisimple anisotropic kernel of $G$	index of $\beta_G(1)$	$r(G)$
$A_2 \times A_2$	3	0
$D_4$	1	$\neq 0$

TABLE 11. Possible invariants of a group  $G$  of type  ${}^1E_6$  that is neither split nor anisotropic.

*Proof (sketch).* As in the proof of the similar result for groups of type  $E_8$  (Lemma 9.1), the list of possible types for the semisimple anisotropic kernel is known. The claim about the index of  $\beta_G(1)$  follows from [Ti 71, p. 211]. As for  $r(G)$ , it belongs to  $H^3(F, \mathbb{Z}/2\mathbb{Z})$ , so we may compute it over a co-2-closure of  $F$ , where  $G$  has trivial Tits algebras; the claim follows as for  $E_8$ . □

REMARK 11.3. We take this opportunity to fill a tiny gap from the classification of possible Tits indexes on p. 311 of [Sp]. That reference omits a proof that a group  $G$  of type  ${}^1E_6$  over a field of characteristic 3 cannot have semisimple anisotropic kernel of type  $A_5$ . To see this, suppose that  $G$  has such a semisimple anisotropic kernel; call it  $M$ . It is isomorphic to  $SL_1(D)$  for a division algebra  $D$  of index 6. The fundamental weight  $\omega_4$  corresponding to the simple root  $\alpha_4$  of  $G$  belongs to the root lattice of  $E_6$ , so the Tits algebra  $\beta_G(\omega_4)$  is zero. On the other

hand, the restriction of  $\omega_4$  to  $M$  has Tits algebra the Brauer class of  $D \otimes D \otimes D$ . By [Ti 71, p. 211], these two Brauer classes are equal, so  $D^{\otimes 3}$  is split and  $M$  is isotropic, a contradiction.

*Proof of Proposition 11.1(2).* It suffices to prove this in the case where  $F$  is 3-special. By Table 11,  $G$  is anisotropic, and remains anisotropic over every cubic extension of  $F$ . It follows from Table 4 that  $G$  cannot contain  $F$ -tori of dimensions 1, 2, or 3, nor dimensions 4 or 5 by symmetry. Consulting [Dy, Table 10], the only possible types for proper semisimple subgroups of maximal rank are  $A_5 \times A_1$  and  $A_2^{\times 3}$ . But the first type contains a 1-dimensional torus, so it cannot be the type of an  $F$ -subgroup of  $G$ .  $\square$

LEMMA 11.4. *Let  $G$  be a group of type  ${}^1E_6$  over a field  $F$  of characteristic  $\neq 2$ . If  $G$  is split by an extension of  $F$  of degree not divisible by 3, then  $G$  contains a subgroup of type  $A_2^{\times 3}$ .*

*Proof.* The hypothesis on  $G$  implies that  $G$  is split—in which case we are done—or  $G$  is isotropic with semisimple anisotropic kernel of type  $D_4$ .

Write  $E_6, D_4, G_2$  for the split simply connected groups of those types. We view  $D_4$  as a subgroup of  $E_6$  via the inclusion from Example 10.4 and  $G_2$  as the subgroup of  $D_4$  consisting of elements fixed by the outer automorphism  $\phi$  of order 3 that cyclically permutes the simple roots  $\alpha_2, \alpha_3, \alpha_5$ . (Recall that the roots of  $E_6$  are all relative to some fixed split maximal torus  $T$ .)

Write  $n$  for an element of  $D_4$  normalizing  $T$  and representing  $-1$  in the Weyl group of  $D_4$ . Replacing  $n$  with  $n\phi(n)\phi^2(n)$ , we may assume that  $n$  belongs to  $G_2$ . Write  $A$  for the group generated by  $n$  and a maximal torus  $T_2 := (T \cap G_2)^0$  in  $G_2$ . The natural map  $H^1(F, A) \rightarrow H^1(F, G_2)$  is surjective, cf. [CS, p. 1060]. Further, the image of  $H^1(F, G_2) \rightarrow H^1(F, \text{Aut}(D_4))$  contains the class of the semisimple anisotropic kernel of  $G$ , because 8-dimensional quadratic forms in  $I^3$  are similar to Pfister forms. Combining these two observations with Tits's Witt-type theorem [Sp, 16.4.2], we deduce that the simply connected cover of  $G$  is obtained by twisting  $E_6$  by a 1-cocycle  $\eta$  with values in  $A$ . By Lemma 10.3,  $\eta$  stabilizes a sub-root-system of type  $A_2^{\times 3}$ , hence the claim.  $\square$

*Proof of Proposition 11.1(1).* Suppose that  $p = 3$ ; otherwise the claim is Lemma 11.4. The adjoint quotient of  $G$  is obtained by twisting the split adjoint  $E_6$  by a 1-cocycle  $\eta$  with values in the normalizer  $N$  of a maximal split torus  $T$  such that the image of  $\eta$  in  $N/T$  is contained in a 3-Sylow subgroup of  $W(E_6)$  of our choosing. It suffices to note that one can find a 3-Sylow in  $W(E_6)$  that normalizes the subsystem  $A_2^{\times 3}$  from 10.1, cf. [C, Lemma 5.8].  $\square$

## 12. Groups of type ${}^2E_6$

Let  $G$  be a group of type  ${}^2E_6$  over  $F$ ; there is a uniquely determined quadratic extension  $K/F$  such that  $G$  is of type  ${}^1E_6$  over  $K$ . The cocenter  $Z^*$  of  $G$  is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$  with a twisted Galois action; an element of  $\text{Gal}(F)$  acts trivially on  $Z^*$  if and only if it restricts to the identity on  $K$ . The number  $n_\chi(1)$  is 27 and the class  $\beta_G(1)$  is represented by a central simple  $K$ -algebra of degree 27. We prove in Theorem 14.1(2) below that there exist such groups  $G$  where  $\beta_G(1)$  has index 27. Proposition 12.1 is an analogue of Proposition 11.1; over a  $p$ -special field for  $p \neq 2$ , every group of type  $E_6$  is inner and we are back in the situation of section 11.

PROPOSITION 12.1. *Let  $G$  be a group of type  ${}^2E_6$  over a field  $F$  of characteristic  $\neq 2$ .*

(1) *If  $F$  is a 2-special field, then  $G$  contains a semisimple  $F$ -subgroup of type  $D_4$ .*

- (2) If  $\beta_G(1)$  has index 27, then  $G$  has no nontrivial, proper, closed connected  $F$ -subgroups other than maximal tori and possibly semisimple subgroups of type  $A_2^{\times 3}$ .

In the case of a 2-special field  $F$ , groups of type  ${}^2E_6$  can still be quite complicated. For example, they can be anisotropic (like the compact real form of  $E_6$ ), which does not occur in the inner case — a fact we exploited in the proof of Proposition 11.1(1).

*Proof.* (1) We have seen that  $G$  is  $K$ -isotropic. The field  $K$  is 2-special, so Table 11 shows that  $G_K$  admits a parabolic subgroup  $P$  of type  $\{1, 6\}$ . By [PIR, p. 383, Lemma 6.17'], the group  $G$  has a reductive  $F$ -subgroup of type  $D_4$ , namely the semisimple part of a Levi subgroup of  $P$ .

(2) Let  $H$  be a nontrivial, proper closed connected  $F$ -subgroup of  $G$ . Over  $K$ ,  $H$  is a maximal torus in  $G$  or is semisimple of type  $A_2^{\times 3}$  by Proposition 11.1. Hence the same is true over  $F$ .  $\square$

### 13. Groups of type $E_7$

In this section we consider groups  $G$  of type  $E_7$ . The cocenter of the simply connected cover of  $G$  is  $\mathbb{Z}/2\mathbb{Z}$ , and the Tits algebra  $\beta_G(1)$  has index dividing  $n_G(1) = 8$ . The element  $r(G)$  belongs to  $H^3(F, \mathbb{Z}/3\mathbb{Z})$ . Our goal is to prove the following:

PROPOSITION 13.1. *Let  $G$  be a group of type  $E_7$  over a field  $F$  of characteristic zero.*

- (1) *If  $F$  is  $p$ -special, then  $G$  contains semisimple subgroups of type  $D_4, A_1^{\times 3}$ , and  $D_4 \times A_1^{\times 3}$ .*
- (2) *If  $\beta_G(1)$  has index 8 and  $r(G)$  is nontrivial, then every proper semisimple subgroup of  $G$  (if there are any) is normalized by a maximal torus of  $G$  and has type  $D_4, A_1^{\times 3}$ , or  $D_4 \times A_1^{\times 3}$ .*

We remark that groups of type  $D_4$  and  $A_1^{\times 3}$  can be almost abelian by Theorem 1.1 and Proposition 3.1. Also, groups  $G$  as in (2) exist; for example, a versal group satisfies the hypothesis by Theorem 8.9 and Lemma 7.5.

*Proof of Proposition 13.1(1).* If the group of type  $E_7$  is split (e.g., if  $p \neq 2, 3$ ), then the result is clear.

If  $F$  is 2-special, we note that the Weyl group of  $E_7$  and of the standard subgroup of type  $D_6 \times A_1$  have the same 2-Sylows, so every  $F$ -group of type  $E_7$  has an  $F$ -subgroup of type  $D_6 \times A_1$ , cf. e.g. [Ga 09, 14.7]. Then it suffices to note that every group of type  $D_6$  contains a subgroup of type  $D_4 \times D_2$ , i.e.,  $D_4 \times A_1^{\times 2}$ .

If  $F$  is 3-special, then every simply connected group of type  $E_7$  is obtained by twisting the split group by a 1-cocycle that normalizes the standard  $D_4$  subgroup [Ga 09, 13.1]. The centralizer of this  $D_4$  subgroup has type  $A_1^{\times 3}$ , corresponding to the sub-root-system spanned by

$$\begin{aligned} \alpha_7, \quad \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \quad \text{and} \\ 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \end{aligned} \tag{13.2}$$

and the conclusion follows.  $\square$

Besides being split or anisotropic, the possible indexes for  $G$  are given by Table 13. The proof is the same as the ones for  $E_8$  and  ${}^1E_6$  given in Lemmas 9.1 and 11.2.

semisimple anisotropic kernel of $G$	index of $\beta_G(1)$	$r(G)$
$E_6$	1	$\neq 0$
$D_6$	divides 4	0
$A_1 \times D_5$	2	0
$A_1 \times D_4$	2	0
$D_4$	1	0
$A_1^{\times 3}$	2	0

TABLE 13. Possible invariants of a group  $G$  of type  $E_7$  that is neither split nor anisotropic.

LEMMA 13.3. Suppose that  $G$  is as in Proposition 13.1(2). Then:

- (1)  $G$  is anisotropic and every extension that splits  $G$  has dimension divisible by 24.
- (2) Let  $K$  be a finite extension of  $F$  such that  $G$  is  $K$ -isotropic.
  - (a) If 3 does not divide  $[K : F]$ , then  $G$  has  $K$ -rank 1 and 8 divides  $[K : F]$ .
  - (b) If 3 divides  $[K : F]$  but 4 does not, then  $G$  has  $K$ -rank 1 and 6 divides  $[K : F]$ .

*Proof.* Since  $\beta_G(1)$  has index 8,  $G$  is anisotropic by Table 13. Further, if  $G$  is split over an extension  $L/F$ , then  $L$  must kill both  $\beta_G(1)$  and  $r(G)$ , hence the dimension  $[L : F]$  is divisible by 8 and 3 respectively. This proves (1).

For (2a), the hypothesis on the dimension implies that  $K$  does not kill  $r(G)$ , hence  $G \times K$  must have semisimple anisotropic kernel  $E_6$  by Table 13.

For (2b), the hypothesis on the dimension implies that  $\beta_G(1) \times K$  has index at least 4 by [GiS, 4.5.11(1)], hence the semisimple anisotropic kernel of  $G \times K$  must have type  $D_6$  by Table 13.  $\square$

The lower bound “divisible by 24” in Lemma 13.3(1) is easily obtained. One might hope to prove a stronger bound, but this is impossible. For any group  $G$  of type  $E_7$ , the greatest common divisor of the dimensions  $[L : F]$ —where  $L$  ranges over extensions of  $F$  splitting  $G$ —divides 24 by [To, 6.1].

*Proof of Proposition 13.1(2).* Because  $G$  is anisotropic and remains so over every quadratic extension of  $F$  by Lemma 13.3(2a), it cannot contain a rank 1 torus nor an  $F$ -subgroup of type  $A_1$ . By symmetry,  $G$  contains no rank 6 tori nor any semisimple subgroup of rank 6.

Over a co-3-closure  $F_3$  of  $F$ , the group  $G$  has rank 1 by Lemma 13.3(2a). But a group of type  $C_3$ ,  $G_2$ ,  $D_n$  for  $n \neq 4$ ,  $A_n$  for  $n = 3, 4, 6, 7$ , or  $A_1^{\times s}$  for  $s \neq 3$  has  $F_3$ -rank  $\geq 2$ . So such a group cannot be an  $F$ -subgroup of  $G$ .

A group of type  $F_4$ ,  $A_2$ , or  $A_5$  has rank  $\geq 2$  over an extension of  $F$  of degree dividing 6, hence it cannot be an  $F$ -subgroup of  $G$  by Lemma 13.3(2b).

Suppose that  $H$  is a semisimple  $F$ -subgroup of  $G$  that is not normalized by a maximal torus of  $G$ . Table 34 on p. 233 of [Dy] gives a list of the possibilities for the type of  $H$ , using the assumption that  $F$  has characteristic zero. All of them have an isotypic component of type  $A_1$ ,  $G_2$ ,  $F_4$ , or  $C_3$ , so they cannot be  $F$ -subgroups of  $G$  by the preceding observations.

Now suppose that  $H$  is a semisimple subgroup of  $G$  of type  $A_2^{\times 2}$ . We claim that its centralizer has semisimple type  $A_2$ . It suffices to prove this over an algebraic closure of  $F$ , where [Dy] says that all subgroup of type  $A_2^{\times 2}$  are conjugate. In particular, we may assume that  $H$  is generated by the root subgroups corresponding to the highest root,  $\alpha_1$ ,  $\alpha_6$ , and  $\alpha_7$  as in the extended

Dynkin diagram

$$\begin{array}{cccccccc}
 & & & & \alpha_2 & & & \\
 & & & & | & & & \\
 \alpha_7 & \alpha_6 & \alpha_5 & & \alpha_4 & \alpha_3 & \alpha_1 & \dots
 \end{array} \tag{13.4}$$

The intersection of the maximal torus  $T$  in  $G$  with  $H$  gives a maximal torus  $S$  in  $H$ . The centralizer  $Z_G(S)$  is reductive with semisimple part generated by the root subgroups for roots  $\alpha_2$  and  $\alpha_4$ . Clearly, these root subgroups centralize  $H$  also, proving the claim. As  $G$  has no  $F$ -subgroups of type  $A_2$ , this is a contradiction, so  $G$  has no subgroups of type  $A_2^{\times 2}$ .

To complete the proof of (2), we consult the list of possible semisimple subalgebras of  $E_7$  in Table 11 in [Dy], and note that all the types other than  $D_4$ ,  $A_1^{\times 3}$ , and  $D_4 \times A_1^{\times 3}$  have rank 6 or have an isotypic component excluded by the previous arguments.  $\square$

13.5. Suppose now that  $G$  is a group of type  $E_7$  over a field  $F$  of characteristic  $\neq 2$ , and that  $G$  contains a semisimple subgroup of type  $D_4 \times A_1^{\times 3}$ . The  $A_1^{\times 3}$  component is isogenous to  $R_{L/F}(\mathrm{SL}(Q))$  for some cubic étale  $F$ -algebra  $L$  and quaternion  $L$ -algebra  $Q$ , and the Tits algebra of  $G$  is Brauer-equivalent to  $\mathrm{cor}_{L/F}[Q]$ . To see this, we examine the center of the  $A_1^{\times 3}$  component in case  $G$  is split. Viewing  $A_1^{\times 3}$  as generated by the roots from (13.2), the nonidentity elements of the center of each copy of  $A_1$  are

$$h_{\alpha_7}(-1), \quad h_{\alpha_2}(-1)h_{\alpha_3}(-1)h_{\alpha_7}(-1), \quad \text{and} \quad h_{\alpha_3}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)$$

respectively, where  $h_{\alpha_i} : \mathbb{G}_m \rightarrow G$  is the cocharacter corresponding to the simple root  $\alpha_i$ . In particular, the fundamental weight  $\omega_7$  of  $G$  restricts to the product map  $\mu_2^{\times 3} \rightarrow \mu_2$  on the center of  $A_1^{\times 3}$ . By [Ti 71, §5], this proves the claim.

The previous paragraph implies that the Tits algebra of every group of type  $E_7$  over a 2-special field is Brauer-equivalent to a tensor product of 3 quaternion algebras. Indeed, it is certainly Brauer-equivalent to  $Q_1 \otimes \mathrm{cor}_{K/F}(Q_2)$  where  $Q_1, Q_2$  are quaternion algebras over  $F$  and a quadratic étale  $F$ -algebra  $K$  respectively. But  $\mathrm{cor}_{K/F}(Q_2)$  has degree 4 and exponent 2, so it is isomorphic to a tensor product of two quaternion algebras.

#### 14. Maximal indexes of Tits algebras

The purpose of this section is extend Theorem 8.9 to include also the most interesting cases where the character  $\chi$  is not fixed by  $\mathrm{Gal}(F)$ . Specifically, we prove:

THEOREM 14.1.

- (1) A superserial group of type  ${}^2A_{n-1}$  for  $n$  odd is isogenous to  $\mathrm{SU}(D, \tau)$  for  $D$  a division algebra of degree  $n$  with unitary involution  $\tau$ . Further,  $\lambda^i D$  has index  $n/\mathrm{gcd}(i, n)$  for  $1 \leq i \leq n$ .
- (2) A superserial group  $G$  of type  ${}^2E_6$  has Tits algebra  $\beta_G(1)$  of index 27.
- (3) A superserial group  $G$  of type  ${}^3D_4$  or  ${}^6D_4$  has Tits algebras  $\beta_G(\chi)$  of index 8 for  $\chi \neq 0$ .

To prove Theorem 1.1, we only need the result for type  ${}^3D_4$ . The result for type  ${}^6D_4$  is known (and we provide here a different proof); it is Proposition 8.1 in [KPS].

The theorem follows easily from Lemma 14.3, stated and proved below.

14.2. We now set up the proof of Lemma 14.3. Fix a root system  $\Phi$ , a set of simple roots  $\Delta$ , a subgroup  $\Gamma$  of  $\mathrm{Aut}(\Delta)$ , and a dominant minuscule weight  $\lambda$ . Write  $G^d$  for the split simply connected group with root system  $\Phi$  and  $(I_\mu, \rho_\mu)$  for the irreducible representation of  $G^d$  with

highest weight  $\mu$ . Define a representation  $(V^d, \rho^d)$  of  $G^d$  by setting

$$V^d := \bigoplus_{\mu \in \Gamma \cdot \lambda} I_\mu \otimes I_\mu^*$$

as a vector space, where  $G^d$  acts on  $I_\mu$  via  $\rho_\mu$  and acts trivially on  $I_\mu^*$ . As a representation of  $G^d$ , we have:

$$(V^d, \rho^d) \cong \bigoplus_{\mu \in \Gamma \cdot \lambda} \bigoplus^{\dim I_\mu} (I_\mu, \rho_\mu).$$

We also define a representation  $\rho^A: \text{Aut}_\Gamma(G^d) \rightarrow \text{GL}(V^d)$  for  $\text{Aut}_\Gamma(G^d)$  as in Def. 6.3. The identity component of  $\text{Aut}_\Gamma(G^d)$  is the adjoint group, and it acts via the natural representation on  $I_\mu$  and  $I_\mu^*$ . (The representations  $\rho_\mu$  and  $\rho_\mu^*$  of  $G^d$  do not factor through the adjoint group, but  $\rho_\mu \otimes \rho_\mu^*$  does.) For each  $\gamma \in \Gamma$ , we have  $\rho_\mu(\gamma^{-1}g) = \text{Int}(a)\rho_{\gamma\mu}(g)$  for some matrix  $a$  depending on  $\gamma$  and  $\mu$ . Combining these matrices with the natural permutation representation on the weights in  $\Gamma \cdot \lambda$ , we find an action of  $\text{Aut}_\Gamma(G^d)$  on  $V^d$ . Furthermore, for  $\alpha \in \text{Aut}_\Gamma(G^d)$  and  $g \in G^d$ , we have:

$$\rho^d(\alpha g) = \text{Int}(\rho^A(\alpha))\rho^d(g).$$

Now fix a 1-cocycle  $z \in Z^1(F, \text{Aut}_\Gamma(G^d))$ . We twist  $G^d$  and  $\rho^d$  by  $z$  to obtain a group  $G = G_z^d$  and a homomorphism  $\rho: G \rightarrow \text{GL}(V^d)_z$ . Hilbert 90 gives an isomorphism between  $\text{GL}(V^d)_z$  and  $\text{GL}(V^d)$ , so  $\rho$  is a representation of  $G$  on  $V^d$ .

More concretely,  $G$  is the same as  $G^d$  over  $F_{\text{sep}}$ , but the Galois group  $\text{Gal}(F)$  acts on  $G(F_{\text{sep}})$  via the action  $\cdot_z$  defined by  $\sigma \cdot_z g := z_\sigma \sigma(g)$ . The 1-cocycle  $\rho^A(z)$  takes values in  $\text{GL}(V^d)$ , and so is  $\sigma \mapsto h^{-1} \sigma h$  for some  $h \in \text{GL}(V)(F_{\text{sep}})$ . We put:  $\rho := \text{Int}(h)\rho^d$ . Then for  $g \in G(F_{\text{sep}})$ , we have:

$$\sigma \rho(g) = \text{Int}(\sigma h)\rho^d(\sigma g) = \text{Int}(h)\text{Int}(\rho^A(z_\sigma))\rho^d(\sigma g) = \text{Int}(h)\rho^d(z_\sigma \sigma g) = \rho(\sigma \cdot_z g),$$

which confirms that  $\rho$  is defined over  $F$  [Bo, AG.14.3].

**LEMMA 14.3.** *Let  $\lambda$  be a minuscule dominant weight of a root system  $\Phi$  with set of simple roots  $\Delta$ . Fix  $\Gamma \subset \text{Aut}(\Delta)$ . If  $|Z^*(F_{\text{sep}})|$  and  $|\Gamma \cdot \lambda|$  are relatively prime, then  $\beta_G(\lambda)$  has index  $n_G(\lambda)$  for every superversal group  $G$  constructed from  $\Delta$  and  $\Gamma$ .*

We define  $\beta_G(\lambda)$  and  $n_G(\lambda)$  where the subscript  $\lambda$  is a weight of  $\Phi$  to be  $\beta_G(\chi)$  and  $n_G(\chi)$  (in the notation of §8), where  $\chi$  is the restriction of  $\lambda$  to the center of  $G$ .

*Proof of Lemma 14.3.* Let  $H$  be a group of type  $\Phi$  over a field  $F$ . The fundamental property we use is that there is a unique irreducible representation  $J_{H,\lambda}$  of  $H$  over  $F$  such that the composition series for  $J_{H,\lambda}$  over an algebraic closure of  $F$  has  $I_\lambda$  as a quotient; this follows from [Ti 71, Th. 7.2]. This representation satisfies

$$\dim J_{H,\lambda} = |\text{Gal}(F) * \lambda| \cdot \text{ind } \beta_H(\lambda) \cdot \dim I_\lambda. \quad (14.4)$$

Twisting the representation  $\rho^d$  of  $G^d$  defined in 14.2 by a superversal torsor gives a representation  $W$  of  $G$ , and the simple quotients in the composition series of  $W$  are copies of  $J_{G,\lambda}$ .

We specialize the superversal group  $G$  to a versal group  $G'$  of inner type defined over a field  $K'$ . The representation  $W$  of  $G$  specializes to a representation  $W'$  of  $G'$ . As  $\lambda$  is fixed by  $\text{Gal}(K')$ , the composition series for  $W'$  over  $K'$  has irreducible quotients various copies of  $J_{G',\mu}$  for  $\mu \in \Gamma \cdot \lambda$ .

We view the irreducible quotients of the composition series as partitioning the weights of  $W$ , equivalently the weights of  $W'$ . Because  $G$ -invariant subspaces of  $W$  specialize to  $G'$ -invariant subspaces of  $W'$ , the partition coming from the composition series under  $G'$  is a refinement of the partition of the composition series under  $G$ . The dimension of  $J_{G',\mu}$  depends only on the  $\text{Aut}(\Delta)$ -orbit of  $\mu$ —by (14.4) and (8.3)—so  $\dim J_{G,\lambda}$  is a multiple of  $\dim J_{G',\lambda}$ . That is,

$$\text{ind } \beta_{G'}(\lambda) \cdot \dim I_\lambda \quad \text{divides} \quad |\Gamma \cdot \lambda| \cdot \text{ind } \beta_G(\lambda) \cdot \dim I_\lambda$$

by (14.4). By Theorem 8.9, the index of  $\beta_{G'}(\lambda)$  is the number  $n_{G'}(\lambda)$ . As multiplication by  $|Z^*(F_{\text{sep}})|$  kills  $\beta_{G'}(\lambda)$  (by (8.1)), every prime dividing  $n_{G'}(\lambda)$  also divides  $|Z^*(F_{\text{sep}})|$  [Gis, 4.5.13(2)]. By the hypothesis on  $|Z^*(F_{\text{sep}})|$ , we deduce that

$$n_{G'}(\lambda) = \text{ind } \beta_{G'}(\lambda) \mid \text{ind } \beta_G(\lambda) \mid n_G(\lambda).$$

Since the numbers  $n_{G'}(\lambda), n_G(\lambda)$  depend only on the root system, they are equal, and this proves the lemma. □

*Proof of Theorem 14.1.* The numbers appearing in Lemma 14.3 are given in the table, where the notation for type  $A$  follows Example 8.8.

type of $G$	$ Z^*(F_{\text{sep}}) $	$\lambda$	$ \Gamma \cdot \lambda $	$n_G(\lambda)$
${}^2A_{n-1}$	$n$	$\omega_i$	2	$n/\text{gcd}(i, n)$
${}^2E_6$	3	minuscule	2	27
${}^3D_4$ or ${}^6D_4$	4	minuscule	3	8

As  $|Z^*(F_{\text{sep}})|$  and  $|\Gamma \cdot \lambda|$  are relatively prime for each of the cases listed in the theorem, the claim follows. □

### 15. Groups of type ${}^3D_4$ or ${}^6D_4$

Let  $G$  be a simple group of type  $D_4$  over  $F$ . The cocenter of its simply connected cover is  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  with a possibly nontrivial Galois action, and  $n_G(\chi)$  is 8 for  $\chi \neq 0$ .

LEMMA 15.1. *Suppose that  $G$  is a group of type  ${}^1D_4, {}^3D_4$  such that  $\beta_G(\chi)$  has index 8 for  $\chi \neq 0$ . Then:*

- (1)  $G$  is anisotropic.
- (2) If  $K/F$  is a finite extension of  $F$  of dimension not divisible by 8, then the  $K$ -rank of  $G$  is  $\leq 1$ .

*Proof.* (1) is known. For  $G$  of type  ${}^3D_4$ , it is [Ga 98, 2.5], and for type  ${}^1D_4$  it is even easier.

For (2), suppose first that  $G$  has type  ${}^1D_4$ , i.e., that  $G$  is isogenous to  $\text{SO}(D, \sigma, f)$  for some central division  $F$ -algebra of degree 8 with quadratic pair such that the even Clifford algebra  $C_0(D, \sigma, f)$  is isomorphic to  $A \times B$  for  $A$  and  $B$  central division  $F$ -algebras of degree 8. By the hypothesis on  $K$ , the index of  $A \otimes K, B \otimes K$ , and  $D \otimes K$  is in all cases at least 2. It follows that the involution  $\sigma$  is not  $K$ -hyperbolic [KMRT, 8.31], which is the claim.

For  $G$  of type  ${}^3D_4$ , there is a unique cubic Galois extension  $L/F$  over which  $G$  is of type  ${}^1D_4$ . The compositum  $KL$  has dimension  $[KL : L]$  dividing  $[K : F]$ , and in particular  $[KL : L]$  is not divisible by 8. Replacing  $F$  and  $K$  with  $L$  and  $KL$ , we are reduced to the type  ${}^1D_4$  case. □

LEMMA 15.2. *Let  $G$  be a group of type  ${}^3D_4$  or  ${}^6D_4$  over a field of characteristic  $\neq 3$ . If  $r(G) \neq 0$ , then  $G$  is anisotropic.*

The hypothesis on the characteristic is here so that  $r(G)$  is defined, cf. Proposition 7.2. Recall from Table 7 that  $r(G)$  is an element of  $H^3(F, \mathbb{Z}/3\mathbb{Z})$ . If  $G$  has type  ${}^1D_4$  (or  ${}^2D_4$ ), then  $r(G)$  is automatically zero.

*Proof of Lemma 15.2.*  $G$  is not quasi-split; suppose it is isotropic. The paper [Ga98] gives a description of  $G$  by Galois descent (which also works in characteristic 2 with only cosmetic changes), and in particular  $G$  is quasi-split by a separable quadratic extension  $K$  of the field. As  $K$  kills  $r(G)$  and the dimension of  $K$  is not divisible by 3,  $r(G)$  is zero.  $\square$

THEOREM 15.3. *Let  $G$  be a group of type  ${}^3D_4$  over a field of characteristic  $\neq 3$ . If the Tits algebras  $\beta_G(\chi)$  have index 8 for nonzero  $\chi$  and  $r(G)$  is not zero, then  $G$  is almost abelian.*

*Proof.* By Lemma 15.2,  $G$  is anisotropic and remains anisotropic over a co-3-closure of  $F$ . It follows that  $G$  does not contain a rank 1  $F$ -torus nor a rank 2  $F$ -torus split by an extension of degree dividing 8. By Lemma 15.1,  $G$  contains no rank 2  $F$ -tori split by an extension of degree dividing 12. By symmetry and Table 4, every nontrivial  $F$ -torus in  $G$  is maximal.

Over an algebraic closure, a proper semisimple subgroup of  $G$  of rank 4 has type  $A_1^{\times 4}$ . An  $F$ -group of this type becomes isotropic over a separable extension of  $F$  of dimension a power of 2, hence by Lemma 15.2 cannot be contained in  $G$ .  $\square$

The hypothesis on  $r(G)$  was used to exclude transfers of groups of type  $A_1$  from a quartic field extension  $L/F$ . These are genuine dangers: The paper [CLM] shows that every isotropic group of type  ${}^3D_4$  or  ${}^6D_4$  contains a subgroup isogenous to  $R_{L/F}(\mathrm{SL}_2)$ .

We close by proving the theorem stated in the introduction.

*Proof of Theorem 1.1.* Let  $G$  be a superversal group of type  ${}^3D_4$  over a field  $F$  of characteristic  $\neq 3$  (itself an extension of some base field  $F_0$ ). Proposition 14.1 gives that  $\beta_G(\chi)$  has index 8 for nonzero  $\chi$ . Because a superversal group is in particular a versal group, we have  $r(G) \neq 0$  by Lemma 7.5. Then  $G$  is almost abelian by Theorem 15.3 and 2.5.

As for type  ${}^6D_4$ , since there are almost abelian groups of type  ${}^3D_4$  defined over a suitable extension of  $F_0$ , Corollary 6.7 implies that the superversal group  $G$  of type  ${}^6D_4$  is almost abelian.  $\square$

*Acknowledgements.* We thank Gopal Prasad for useful conversations about the global field case. We also thank the referee, whose comments improved the exposition.

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