Orthogonal involutions on algebras of degree 16 and the Killing form of E_8

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With an appendix by Kirill Zainoulline

Abstract. We exploit various inclusions of algebraic groups to give a new construction of groups of type E_8, determine the Killing forms of the resulting E_8's, and define an invariant of central simple algebras of degree 16 with orthogonal involution "in I^3", equivalently, groups of type D_8 with a half-spin representation defined over the base field. The determination of the Killing form is done by restricting the adjoint representation to various twisted forms of PGL_2.

An appendix by Kirill Zainoulline contains a type of "index reduction" result for groups of type D.

The first part of this paper (§§1–6) extends the Arason invariant e_3 for quadratic forms in I^3 to central simple algebras (A, σ) "in I^3" (this term is defined in §1) where A has degree 16 or has a hyperbolic involution. (The first case corresponds to simple linear algebraic groups of type D_8 with a half-spin representation defined over the base field.) The invariant e_3 detects whether (A, σ) is generically Pfister, see Corollary 2.9 below. We remark that the paper [**BPQ**] appears to rule out the existence of such an invariant by a counterexample. Our invariant exists exactly in the cases where their counterexample does not apply; surprisingly, this includes some interesting cases. The proofs in this part are not difficult, but we include this material to provide background and context for the later results. Proposition 1.4 generalizes the Arason-Pfister Hauptsatz for quadratic forms of dimension < 16, and depends on a result of Kirill Zainoulline presented in Appendix A.

The real work begins in the second part of the paper (§§7–10), where we use the inclusion $\mathrm{PGL}_2 \times \mathrm{PSp}_8 \subset \mathrm{PSO}_8$ to give a formula for the Arason invariant in case (A, σ) can be written as a tensor product $(Q, ^-) \otimes (C, \gamma)$, where $(Q, ^-)$ is a quaternion algebra with its canonical symplectic involution and C has degree 8.

We apply the preceding results in the third part of the paper (§§11–16) to studying algebraic groups of type E_8. These groups are poorly understood relative to other types of simple algebraic groups. For example, the cohomological invariants

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of the split E_8—in the sense of [**Ga 09**, p. 4]—are not classified, and we do not even know if there are any nontrivial invariants of degree ≥ 4.

The principal construction of groups of type E_8 in the literature is Tits's from [**Ti 66a**], see 11.6 below. We give a different construction in (11.2); it takes as input four quaternion algebras and an element of $k^\times/k^{\times 2}$. We compute the Rost invariant, Tits index (in some cases), and Killing form of the resulting E_8's, see Th. 9.1, Prop. 12.1, and Th. 15.2. We compute the Killing form by branching to subgroups of type A_1, which is somewhat cleaner than computations of other Killing forms in the literature. (The Rost invariant and Killing form for the E_8's arising from Tits's construction have been known for a long time; see 11.6 and Example 13.2. Computing the Tits index in general can be difficult; the presumably simpler case of outer type E_6 took all of the paper [**GPe**].) The E_8's arising from our construction are an interesting class. They are uncomplicated enough to be tractable, yet include all E_8's over every number field.

The motivation for studying this construction comes from the problem of classifying groups of type E_8 over an arbitrary field. This problem is currently out of reach, but we can hope to solve the presumably much easier problem of classifying groups of type E_8 that lie in the kernel of the Rost invariant and are defined over a field k all of whose separable extensions have degree a power of 2. This class of E_8's is nontrivial: for k the real numbers, it contains both the split and the compact forms of E_8, see Example 13.2. If the converse of the Pfister Factor Conjecture holds, then our construction produces all E_8's in this class, see Th. 16.2 for details.

Contents

Notation and conventions. We work over a field k of characteristic $\neq 2$. Throughout the paper, (A, σ) denotes a central simple k-algebra with orthogonal involution.

We often write $^-$ for the canonical symplectic involution on a quaternion algebra; it will be clear by context which quaternion algebra is intended. Similarly, we write hyp for a hyperbolic involution; context again will make it clear whether symplectic or orthogonal is intended.

For g in a group G, we write $\mathrm{Int}(g)$ for the automorphism $x \mapsto gxg^{-1}$.

General background on algebras with involution can be found in [**KMRT**]. For the Rost invariant, see [**Mer 03**] or [**KMRT**, §31].

Part I. Extending the Arason invariant to orthogonal involutions

1. I^n

Definition 1.1. Let (A, σ) be a central simple algebra with orthogonal involution over a field k of characteristic $\neq 2$. The function field k_A of the Severi-Brauer variety of A splits A, hence over k_A the involution σ is adjoint to a quadratic form q_σ determined up to similarity. As an abbreviation, we say that (A, σ) is *in $I^n k$* (or simply "in I^n") if q_σ belongs to $I^n k_A$, the n-th power of the fundamental ideal in the Witt ring of k_A.

This definition is "compatible with scalar extension". That is, *If (A, σ) is in $I^n k$ and L is an extension of k, then $(A \otimes L, \sigma \otimes \mathrm{Id})$ is in $I^n L$.* To see this, we write X for the Severi-Brauer variety of A and note that $L(X)$ is obviously an extension of $k(X)$.

Example. If A is split over k, then the quadratic form q_σ can be chosen to be defined over k. The extension k_A/k is purely transcendental, so $q_\sigma \otimes k_A$ belongs to $I^n k_A$ if and only if q_σ belongs to $I^n k$. That is, *(A, σ) is in $I^n k$ if and only if q_σ is in $I^n k$.*

This first part of the paper is concerned with cohomological invariants of (A, σ) in case (A, σ) is in I^3, especially when A has degree 16. For context, we give some properties of algebras with involution in I^n for small n.

Examples 1.2. A quadratic form is in I if and only if its dimension is even, so obviously an algebra with involution (A, σ) is in I if and only if the degree of A is even. More interestingly, we have:

(1) *(A, σ) is in I^2 if and only if it is in I and the discriminant of σ is zero in $k^\times/k^{\times 2}$.* This follows from the fact that k is algebraically closed in k_A, so the discriminant is a square in k if and only if it is a square in k_A.

In particular, if (A, σ) is in I^2, then A has even degree and the even Clifford algebra $C(A, \sigma)$ is isomorphic to a product $C_+ \times C_-$ of two central simple algebras. We recall the fundamental relation $[C_+] - [C_-] = [A]$ in the Brauer group from [**KMRT**, 9.12] or [**Ti 71**, 6.2].

(2) *(A, σ) is in I^3 if and only if it is in I^2 and one component of the even Clifford algebra $C(A, \sigma)$ is split.* If A is split, this is known as Merkurjev's

Theorem [**Mer 81**]. Otherwise, suppose A is not split and (A, σ) is in I^2. Then (A, σ) is in I^3 if and only if k_A splits both C_+ and C_-. But the only nonzero Brauer class killed by k_A is $[A]$ and only one of C_+ and C_- can be Brauer-equivalent to $[A]$ by the fundamental relation, hence the claim.

(3) Suppose that (A, σ) is Witt-equivalent to (A', σ') in the sense of [**DLT**]. Then (A, σ) is in I^n if and only if (A', σ') is in I^n.

(4) We say that (A, σ) is *generically Pfister* if q_σ is similar to a Pfister form. Suppose that $\deg A = 2^n$. Then (A, σ) *is in* I^n *if and only if* (A, σ) *is generically Pfister*, because the only quadratic forms of dimension 2^n in I^n are scalar multiples of n-Pfister forms [**Lam**, X.5.6].

(5) Suppose that $\deg A = 2^n$ with $n \geq 1$. If (A, σ) is *completely decomposable* (i.e., isomorphic to a tensor product of quaternion algebras with orthogonal involution), then (A, σ) is generically Pfister by [**Bech**], hence is in I^n.

The preceding examples show that the property of an algebra being in I^n for $n \leq 3$ can be detected by invariants defined over k, without going up to the generic splitting field k_A. Below we construct an invariant that detects whether (A, σ) belongs to I^4 for A of degree 16.

QUESTION 1.3. The converse to (5) holds for $n = 1$ (trivial), $n = 2$ [**KPS**], and $n = 3$ [**KMRT**, 42.11]. Does the converse also hold for $n = 4$? That is, *does generically Pfister and degree* 16 *imply completely decomposable?* The answer is "yes" if A has index 1 (obvious) or 2 [**Bech**, Th. 2]. We return to this question in §16 below.

PROPOSITION 1.4 ("Arason-Pfister"). *Suppose that* (A, σ) *is in* I^n *for some* $n \geq 1$ *and* $\deg A < 2^n$. *If* $n \leq 4$, *then* σ *is hyperbolic (and* A *is not a division algebra).*

PROOF. The case where A has index 1 is the Arason-Pfister Hauptsatz for quadratic forms [**Lam**, X.5.1]. Otherwise, the Hauptsatz implies that σ is hyperbolic over k_A. If A has index 2 we are done by [**PSS**, Prop. 3.3], and if $\deg A / \operatorname{ind} A$ is odd we are done by Proposition A.1.

The remaining case is where A has degree 8 and index 4, (A, σ) is in I^3, and one component of its even Clifford algebra is split. The canonical involution on the even Clifford algebra restricts to be orthogonal on the split component; it is adjoint to an 8-dimensional quadratic form ϕ that has trivial discriminant because $\deg A$ is divisible by 8 [**Ga 01a**, 1.3]. The involution σ is hyperbolic over k_A by the index 1 case, hence ϕ is also hyperbolic over k_A [**Ga 01a**, 1.1]. Triality as in [**KMRT**, §42] shows that A is Brauer-equivalent to the full Clifford algebra of ϕ, so ϕ is isotropic over the base field k by [**Lag**, Th. 4]. It follows that σ is hyperbolic over k [**Ga 01a**, 1.1]. □

The algebras of degree 8 in I^3 are completely decomposable by [**KMRT**, 42.11]. For degree 10, we have the following nice observation pointed out by Jean-Pierre Tignol:

LEMMA 1.5. *If* (A, σ) *is in* I^3 *and* $\deg A \equiv 2 \bmod 4$, *then* A *is split.*

In particular, if (A, σ) is of degree 10 and in I^3, then A is split, hence σ is isotropic by Pfister, see [**Lam**, XII.2.8] or [**Ga 09**, 17.8].

Proof of Lemma 1.5. We follow the notation of Example 1.2(1). Because the degree of A is congruent to 2 mod 4, the Brauer class $[C_+]$ of one component of the even Clifford algebra satisfies $2[C_+] = [A]$, see [**KMRT**, 9.15] or [**Ti 71**, 6.2]. But (A, σ) belongs to I^3, so $[C_+]$ is 0 or $[A]$. Hence $[A] = 0$. □

Algebras (A, σ) in I^3 of degree 12 are described in [**GQ 09**].

For (A, σ) in I^3 of degree 14, the algebra A is split by Lemma 1.5, hence σ is adjoint to a quadratic form in I^3. These forms have been described by Rost, see [**R**] or [**Ga 09**, 17.8].

For (A, σ) in I^3 and of degree ≥ 16, the main question to ask is: *How to tell if (A, σ) is in I^4?* We address that question in Corollary 2.8 below.

Remark 1.6 (I $\Rightarrow$ H). In addition to the generically Pfister and completely decomposable algebras with involution, another interesting class of involutions are the so-called $I \Rightarrow H$ involutions. We say that a central simple k-algebra A with orthogonal involution σ *has* $I \Rightarrow H$ if the degree of A is 2^n for some $n \geq 1$, and for every extension K/k over which σ is isotropic, the involution σ is actually K-hyperbolic. If (A, σ) has I $\Rightarrow$ H, then (A, σ) is generically Pfister, see [**BPQ**]. Conversely, if $n \leq 4$ and (A, σ) is generically Pfister, then (A, σ) has I $\Rightarrow$ H by the arguments in the proof of Proposition 1.4.

2. Extending the Arason invariant

Recall that for every $n \geq 0$ there is an additive map $e_n \colon I^n k \to H^n(k, \mathbb{Z}/2\mathbb{Z})$ such that

$$e_n(\langle \alpha \rangle q) = e_n(q) \quad \text{for } \alpha \in k^\times \text{ and } q \in I^n k \tag{2.1}$$

and

$$\ker e_n = I^{n+1} k. \tag{2.2}$$

We have the examples:

- e_0 gives the dimension mod 2.
- e_1 is the (signed) discriminant denoted by $d_\pm$ in [**Lam**].
- e_2 is the Clifford invariant defined in [**Lam**]. We already used property (2.2) for e_2 in Example 1.2(2).
- e_3 is the Arason invariant defined in [**A**].

The existence of the e_n's for higher n is known by work of Merkurjev, Suslin, Rost, etc., culminating in [**OVV**].

If one is given a central simple algebra with orthogonal involution (A, σ) such that A is split, then σ is adjoint to a quadratic form q_σ determined up to similarity. By (2.1), the value of $e_n(q_\sigma)$ depends only on σ and not on the choice of q_σ. One might ask if there is some way to define $e_n(A, \sigma)$ for (A, σ) in I^n so that $e_n(A, \sigma)$ equals $e_n(q_\sigma)$ if A is split. For $n = 0, 1, 2$, this is standard:

- $e_0(A, \sigma)$ is the degree of A mod 2.
- $e_1(A, \sigma)$ is the discriminant of σ defined in [**KMRT**, 7.2].
- $e_2(A, \sigma)$ is the class of one component of the even Clifford algebra of (A, σ) in $H^2(k, \mathbb{Z}/2\mathbb{Z})/[A]$.

Note that e_2 takes values not in $H^2(k, \mathbb{Z}/2\mathbb{Z})$ but rather in $H^2(k, \mathbb{Z}/2\mathbb{Z})$ modulo the kernel of the restriction map $H^2(k, \mathbb{Z}/2\mathbb{Z}) \to H^2(k_A, \mathbb{Z}/2\mathbb{Z})$.

Now suppose that (A, σ) is in I^3. We look for an element

$$e_3(A, \sigma) \in H^3(k, \mathbb{Z}/4\mathbb{Z})/E(A) \tag{2.3}$$

for $E(A) := \ker(H^3(k, \mathbb{Z}/4\mathbb{Z}) \to H^3(k_A, \mathbb{Z}/4\mathbb{Z}))$, such that

(2.4) If K/k splits A, then $\mathrm{res}_{K/k}\, e_3(A, \sigma)$ is the Arason invariant $e_3(q_{\sigma \otimes K})$ in $H^3(K, \mathbb{Z}/4\mathbb{Z})$.

(Some care must be taken in (2.4), as the Arason invariant $e_3(q_{\sigma\otimes K})$ naturally lives in $H^3(K, \mathbb{Z}/2\mathbb{Z})$. But the inclusion $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$ identifies $H^3(K, \mathbb{Z}/2\mathbb{Z})$ with the 2-torsion in $H^3(K, \mathbb{Z}/4\mathbb{Z})$ by [**MS**], and in this way we view $e_3(q_{\sigma\otimes K})$ as an element of $H^3(K, \mathbb{Z}/4\mathbb{Z})$. Further, as K splits A, there is a k-place from k_A to K, hence $\mathrm{res}_{K/k}$ kills $E(A)$, $\mathrm{res}_{K/k}\, e_3(A, \sigma)$ is a well-defined element of $H^3(K, \mathbb{Z}/4\mathbb{Z})$, and we may compare the two elements of $H^3(K, \mathbb{Z}/4\mathbb{Z})$.)

Clearly, properties (2.3) and (2.4) uniquely determine $e_3(A, \sigma)$ *if such an element exists*. Given such an element $e_3(A, \sigma)$, we can define an element $e_3\,[(A, \sigma) \otimes L]$ for every extension L/k by setting:

$$e_3\,[(A, \sigma) \otimes L] := \mathrm{res}_{L/k}(e_3(A, \sigma)). \tag{2.5}$$

This element satisfies (2.3) and (2.4), with k replaced by L.

But does an element $e_3(A, \sigma)$ satisfying (2.3) and (2.4) exist? If A is split, the answer is of course "yes". If A has index 2, then the element $e_3\,[(A, \sigma) \otimes k_A] \in H^3(k_A, \mathbb{Z}/4\mathbb{Z})$ is unramified [**Ber**, Prop. 9] and so descends to define an element $e_3(A, \sigma)$ as above by [**KRS**, Prop. A.1]. However, an element $e_3(A, \sigma)$ need not exist if A has degree 8 and is division as Theorem 3.9 in [**BPQ**] shows. We have:

THEOREM 2.6. *Suppose* $\operatorname{ind} A \le 2$ *or* $2 \operatorname{ind} A$ *divides* $\deg A$ *or* $\deg A = 16$. *Then there exists an* $e_3(A, \sigma)$ *as in* (2.3) *and* (2.4).

To illustrate the cases covered by the proposition, we note that for A of even degree between 8 and 16, the only omitted cases are where A has degree 8 and index 8 or A has degree 12 and index 4. These cases are genuinely forbidden by [**BPQ**] and the following example, which extends slightly the reasoning in [**BPQ**].

EXAMPLE 2.7. Fix a field k_0 and an algebra with orthogonal involution (A, σ) in I^3 over k_0, where A has degree 12 and index 4. By extending scalars to various function fields of quadrics as in [**Mer 92**], we can construct an extension k/k_0 such that $H^3(k, \mathbb{Z}/4\mathbb{Z})$ is zero and $A \otimes k$ still has index 4. (More precisely, A is isomorphic to 3-by-3 matrices over a division algebra D that is a tensor product of two quaternion algebras. The proof of Theorem 4 in [**Mer 92**] gives a k such that $D \otimes k$ is division and $H^3(k, \mathbb{Z}/2\mathbb{Z})$ is zero. But $H^3(k, \mathbb{Z}/2\mathbb{Z})$ is the 2-torsion in $H^3(k, \mathbb{Z}/4\mathbb{Z})$, a group that is itself killed by 4. So $H^3(k, \mathbb{Z}/4\mathbb{Z})$ is also zero.)

We claim that there is no element $e_3(A, \sigma)$ satisfying (2.3) and (2.4). Indeed, by (2.3), such an $e_3(A, \sigma)$ would be zero. By (2.4) and (2.2), the quadratic form q_σ in $I^3 k_A$ belongs to $I^4 k_A$, hence is hyperbolic by the Arason-Pfister Hauptsatz. That is, σ is hyperbolic over k_A, but this is impossible by Proposition A.1.

By adding hyperbolic planes, this example and the one from [**BPQ**] show that there exist $(A, \sigma) \in I^3$

- of index 8 and degree $8 + 16\ell$
- of index 4 and degree $12 + 8\ell$

for all $\ell \geq 0$ such that no element $e_3(A,\sigma)$ satisfies (2.3) and (2.4).

As for the proof of Theorem 2.6, the case of index ≤ 2 was treated in [**Ber**], as outlined above. In the remaining two cases, we define invariants e_3^{hyp} and e_3^{16} in §3 and §6 respectively that take values in $H^3(k,\mathbb{Z}/4\mathbb{Z})/[A]\cdot H^1(k,\boldsymbol{\mu}_2)$. Clearly, $[A]\cdot H^1(k,\boldsymbol{\mu}_2)$ is contained in $E(A)$,[a] and we define $e_3(A,\sigma)$ to be the image of $e_3^{\mathrm{hyp}}(A,\sigma)$ or $e_3^{16}(A,\sigma)$ in $H^3(k,\mathbb{Z}/4\mathbb{Z})/E(A)$. Property (2.4) is proved in Examples 3.5 and 6.3 below.

Corollary 2.8. *An algebra with involution (A,σ) as in Theorem 2.6 belongs to I^4 if and only if $e_3(A,\sigma)$ is zero.*

Proof. (A,σ) belongs to I^4 if and only if q_σ is in I^4k_A if and only if $e_3(A,\sigma)$ is killed by k_A. □

Corollary 2.9. *An algebra $(A,\sigma)\in I^3$ of degree 16 is generically Pfister if and only if $e_3(A,\sigma)$ is zero.*

Proof. Combine the previous corollary and Example 1.2(4). □

3. Invariant $e_3^{\mathrm{hyp}}(A,\sigma)$

Suppose that (A,σ) is in I^3, and 2 times the index of A divides the degree of A, i.e., there is a hyperbolic (orthogonal) involution "hyp" defined on A. We now define an element $e_3^{\mathrm{hyp}}(A,\sigma)\in H^3(k,\mathbb{Z}/4\mathbb{Z})/[A]\cdot H^1(k,\boldsymbol{\mu}_2)$ that agrees with the Arason invariant of (A,σ) in case A is split.

We may assume that A has degree ≥ 8; otherwise, σ is hyperbolic by Prop. 1.4 and we set $e_3^{\mathrm{hyp}}(A,\sigma)=0$. We assume further that 4 divides the degree of A; otherwise, A has index at most 2 and we set $e_3^{\mathrm{hyp}}(A,\sigma)$ to be the invariant defined by Berhuy. These two assumptions imply that $\mathrm{Spin}(A,\mathrm{hyp})$ is a simple algebraic group of type D_ℓ for ℓ even and ≥ 4.

Put Z for the center of $\mathrm{Spin}(A,\mathrm{hyp})$; it is isomorphic to $\boldsymbol{\mu}_2\times\boldsymbol{\mu}_2$.

Lemma 3.1. *The sequence*

$$H^1(k,Z)\longrightarrow H^1(k,\mathrm{Spin}(A,\mathrm{hyp}))\xrightarrow{\ q\ } H^1(k,\mathrm{Aut}(\mathrm{Spin}(A,\mathrm{hyp})))$$

is exact, and the fibers of q are the $H^1(k,Z)$-orbits in $H^1(k,\mathrm{Spin}(A,\mathrm{hyp}))$.

If one replaces $\mathrm{Aut}(\mathrm{Spin}(A,\mathrm{hyp}))$ with its identity component, then the lemma is obviously true.

Sketch of proof of Lemma 3.1. Given a 1-cocycle with values in the group $\mathrm{Spin}(A,\mathrm{hyp})$, we write G for the group $\mathrm{Spin}(A,\mathrm{hyp})$ twisted by the 1-cocycle. The center of G is canonically identified with Z, and we want to show that the sequence

$$H^1(k,Z)\longrightarrow H^1(k,G)\xrightarrow{\ q\ } H^1(k,\mathrm{Aut}(G)) \tag{3.2}$$

is exact.

Suppose that $\hat{g}\in H^1(k,G)$ is killed by q, i.e., the twisted group $G_{q(\hat{g})}$ is isomorphic to G. By the exactness of the sequence

$$1\longrightarrow \mathrm{Aut}(G)^\circ\longrightarrow \mathrm{Aut}(G)\longrightarrow \mathrm{Aut}(\Delta)\longrightarrow 1$$

[a]This inclusion is proper for some algebras A of index ≥ 8 by [**Pey**] and [**Ka 98**, 5.1].

for Δ the Dynkin diagram of G [**Sp**, §16.3], the image γ of $\hat{g}$ in $H^1(k, \mathrm{Aut}(G)^\circ)$ is also the image of some $\pi \in \mathrm{Aut}(\Delta)(k)$. The element π acts on Z, hence on $H^2(k, Z)$, and since $G_{q(\hat{g})}$ is isomorphic to G,

(3.3) The automorphism π fixes the Tits class of G in $H^2(k, Z)$.

We now show that π is in the image of $\mathrm{Aut}(G)(k)$; we assume that $\pi \neq 1$. We write G as $\mathrm{Spin}(A, \tau)$ for some (A, τ) in I^3. The even Clifford algebra of (A, τ) is Brauer-equivalent to $A \times k$. If $\deg A \neq 8$, then π has order 2 and (3.3) implies that A is isomorphic to $M_{2\ell}(k)$ [**KMRT**, p. 379], i.e., A is split, so a hyperplane reflection in $\mathrm{Aut}(G)(k)$ maps to π. If $\deg A = 8$, then $\mathrm{Aut}(\Delta)$ is the symmetric group on 3 letters, but similar reasoning applies; the case where π has order 3 is [**KMRT**, 35.6].

Since π is in the image of $\mathrm{Aut}(G)(k)$, the element $\gamma \in H^1(k, \mathrm{Aut}(G)^\circ)$ is zero, hence $\hat{g}$ comes from $H^1(k, Z)$. This proves that (3.2) is exact. □

There is a class $\eta' \in H^1(k, \mathrm{Spin}(A, \mathrm{hyp}))$ that maps to the class of $\mathrm{Spin}\,(A, \sigma)$ in $H^1(k, \mathrm{Aut}(\mathrm{Spin}(A, \mathrm{hyp})))$, and Lemma 3.1 says that η' is determined up to the action of $H^1(k, Z)$.

We put:

$$e_3^{\mathrm{hyp}}(A, \sigma) := r_{\mathrm{Spin}(A,\mathrm{hyp})}(\eta') \quad \in \frac{H^3(k, \mathbb{Z}/4\mathbb{Z})}{[A] \cdot H^1(k, \boldsymbol{\mu}_2)}\ , \tag{3.4}$$

where r denotes the Rost invariant. The main result of [**MPT**] says that the image of $H^1(k, Z)$ under $r_{\mathrm{Spin}(A,\mathrm{hyp})}$ is $[A] \cdot H^1(k, \boldsymbol{\mu}_2)$. Combining this and the fact that the Rost invariant is compatible with the action of the center [**Ga 01b**, 7.1] shows that $e_3^{\mathrm{hyp}}(A, \sigma)$ depends only on (A, σ) and not on the choice of η'.

Example 3.5. If A is split, then σ is adjoint to a quadratic form q_σ and $e_3^{\mathrm{hyp}}(A, \sigma)$ is the Arason invariant of q_σ by [**KMRT**, p. 436].

4. I^3 and D_{2n}

Write Spin_{4n} for the split simply connected group of type D_{2n}. Its center is $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. Up to isomorphism, Spin_{4n} has four quotients: itself, SO_{4n}, the adjoint group PSO_{4n}, and one other that we call a half-spin group[b] and denote by HSpin_{4n}. We are interested in it because of the following result:

Lemma 4.1. *The image of $H^1(k, \mathrm{HSpin}_{4n})$ in $H^1(k, \mathrm{PSO}_{4n})$ classifies pairs (A, σ) of degree $4n$ in I^3.*

The algebras of (A, σ) in I^3 with degree divisible by 4 are in some sense the most interesting ones. If the degree is not divisible by 4, then A is split by Lemma 1.5, and classifying such (A, σ) is a problem in quadratic form theory.

Proof of Lemma 4.1. The proof can be summarized by writing: Combine pages 409 and 379 in [**KMRT**].

We identify the groups Spin_{4n} and PSO_{4n} with the corresponding groups for the split central simple algebra B of degree $4n$ with hyperbolic orthogonal involution τ. Fix a labeling $C_+ \times C_-$ for the even Clifford algebra $C(B, \tau)$. Write π_+ for the projection $C(B, \tau) \to C_+$ and HSpin_{4n} for the image of Spin_{4n} in C_+ under π_+.

[b] Bourbaki writes "semi-spin" in [**Bou**].

Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 & \longrightarrow & \mathrm{Spin}_{4n} & \longrightarrow & \mathrm{PSO}_{4n} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \pi_+} & & \Vert & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathrm{HSpin}_{4n} & \longrightarrow & \mathrm{PSO}_{4n} & \longrightarrow & 1
\end{array}
\tag{4.2}
$$

It induces a commutative diagram with exact rows:

$$
\begin{array}{ccccc}
H^1(k, \mathrm{Spin}_{4n}) & \longrightarrow & H^1(k, \mathrm{PSO}_{4n}) & \longrightarrow & H^2(k, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \\
\downarrow{\scriptstyle \pi_+} & & \Vert & & \downarrow{\scriptstyle \pi_+} \\
H^1(k, \mathrm{HSpin}_{4n}) & \longrightarrow & H^1(k, \mathrm{PSO}_{4n}) & \longrightarrow & H^2(k, \boldsymbol{\mu}_2)
\end{array}
\tag{4.3}
$$

As in [**KMRT**, p. 409], the set $H^1(k, \mathrm{PSO}_{4n})$ classifies triples (A, σ, ϕ) where A has degree $4n$, the involution σ is orthogonal with trivial discriminant, and ϕ is a k-algebra isomorphism $Z(C(A, \sigma)) \xrightarrow{\sim} Z(C_+ \times C_-)$. We view ϕ as a labeling of the components of the even Clifford algebra of (A, σ) as $+$ and $-$. The image of such a triple (A, σ, ϕ) in $H^2(k, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$ is the Tits class of $\mathrm{Spin}(A, \sigma, \phi)$, and it follows from [**KMRT**, p. 379] and the commutativity of (4.3) that the image of (A, σ, ϕ) in $H^2(k, \boldsymbol{\mu}_2)$ is $C_+(A, \sigma)$. So (A, σ, ϕ) is in the image of $H^1(k, \mathrm{HSpin}_{4n})$ if and only if $C_+(A, \sigma)$ is split.

We have proved that for every (A, σ) of degree $4n$ in I^3, there is some triple (A, σ, ϕ) in the image of $H^1(k, \mathrm{HSpin}_{4n}) \to H^1(k, \mathrm{PSO}_{4n})$. Suppose now that (A, σ, ϕ) and (A, σ, ϕ') are in the image of $H^1(k, \mathrm{HSpin}_{4n})$; we will show they are equal; we may assume that $\phi \neq \phi'$. If A is split, then a hyperplane reflection gives an isomorphism between (A, σ, ϕ) and (A, σ, ϕ'), and we are done. If A is nonsplit, then $C_-(A, \sigma)$ (with numbering given by ϕ) is nonsplit and it is impossible that (A, σ, ϕ') is in the image of $H^1(k, \mathrm{HSpin}_{4n})$. This concludes the proof. $\square$

5. $\mathrm{HSpin}_{16} \subset E_8$

Write E_8 for the split algebraic group of that type. We view it as generated by homomorphisms $x_\varepsilon \colon \mathbb{G}_a \to E_8$ as ε varies over the root system E_8 of that type, as in [**St**]. The root system D_8 is contained in E_8, as can be seen from the Dynkin diagrams as in Figure 5A. Symbolically, we can see the inclusion as follows. Fix sets of simple roots $\delta_1, \ldots, \delta_8$ of D_8 and $\varepsilon_1, \ldots, \varepsilon_8$ of E_8, numbered as in Figure 5A, where $\tilde{\varepsilon}$ of the highest root of E_8. The inclusion of D_8 in E_8 is given by the following table:

(5.1)

D_8 root	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8
E_8 root	$-\tilde{\varepsilon}$	ε_8	ε_7	ε_6	ε_5	ε_4	ε_2	ε_3

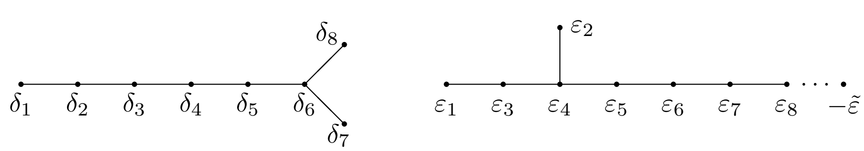


Figure 5A. Dynkin diagram of D_8 and extended Dynkin diagram of E_8

The subgroup of E_8 generated by the x_ε's for $\varepsilon \in \mathsf{D}_8$ is a subgroup of type D_8. This is standard, see e.g. [**BdS**]. Moreover, the subgroup of type D_8 is HSpin_{16}. This can be seen by root system computations as in [**Ti 90**, §1.7] or with computations in the centers as in [**GQ 08**].

It follows from Table 5.1 (and is asserted in [**Dy**]), that:

(5.2) *The composition* $\mathrm{Spin}_{16} \to \mathrm{HSpin}_{16} \to E_8$ *has Rost multiplier* 1.

That is, the composition $H^1(k, \mathrm{Spin}_{16}) \to H^1(k, E_8) \xrightarrow{r_{E_8}} H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ is $r_{\mathrm{Spin}_{16}}$, the usual Rost invariant for Spin_{16}.

5.3. Let $\eta \in H^1(k, \mathrm{HSpin}_{16})$ map to the class of (A, σ) in $H^1(k, \mathrm{PSO}_{16})$, cf. Lemma 4.1. We write $\mathrm{HSpin}\,(A,\sigma)$ for the group HSpin_{16} twisted by η; it is the image of $\mathrm{Spin}\,(A,\sigma)$ in a split component of $C(A,\sigma)$. We find homomorphisms

$$\mathrm{Spin}\,(A,\sigma) \to \mathrm{HSpin}\,(A,\sigma) \to (E_8)_\eta.$$

The center of $\mathrm{HSpin}\,(A,\sigma)$ is a copy of $\boldsymbol{\mu}_2$, and we compute the image of an element $c \in k^\times/k^{\times 2}$ under the composition

$$k^\times/k^{\times 2} = H^1(k, \boldsymbol{\mu}_2) \to H^1(k, \mathrm{HSpin}\,(A,\sigma)) \to H^1(k, (E_8)_\eta) \xrightarrow{r_{(E_8)_\eta}} H^3(k, \mathbb{Q}/\mathbb{Z}(2)).$$

The center of $\mathrm{Spin}\,(A,\sigma)$ is $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, and it maps onto the center of $\mathrm{HSpin}\,(A,\sigma)$ via the map π_+ from the proof of Lemma 4.1. The induced map $H^1(k, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \to H^1(k, \boldsymbol{\mu}_2)$ is obviously surjective, so there is some $\gamma \in H^1(k, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$ such that $\pi_+(\gamma) = (c)$. By (5.2), we have:

$$r_{(E_8)_\eta}(c) = r_{\mathrm{Spin}\,(A,\sigma)}(\gamma),$$

which is $\pi_+(\gamma) \cdot [A]$ by [**MPT**], i.e., $(c) \cdot [A]$.

6. Invariant $e_3^{16}(A,\sigma)$ for algebras of degree 16 in I^3

Let (A,σ) be of degree 16 in I^3. Fix a class $\eta \in H^1(k, \mathrm{HSpin}_{16})$ that maps to the class of (A,σ) in $H^1(k, \mathrm{PSO}_{16})$. (Here we are using Lemma 4.1 to know that there is a uniquely determined element in $H^1(k, \mathrm{PSO}_{16})$.) Consider the image $r_{E_8}(\eta)$ of η under the map

$$H^1(k, \mathrm{HSpin}_{16}) \to H^1(k, E_8) \xrightarrow{r_{E_8}} H^3(k, \mathbb{Q}/\mathbb{Z}(2)).$$

Since (A,σ) is killed by an extension of k of degree a power of 2, the same is true for η, hence also for $r_{E_8}(\eta)$. As $r_{E_8}(\eta)$ is 60-torsion, we conclude that $r_{E_8}(\eta)$ is 4-torsion, i.e., $r_{E_8}(\eta)$ belongs to $H^3(k, \mathbb{Z}/4\mathbb{Z})$. We define

$$e_3^{16}(A,\sigma) := r_{E_8}(\eta) \quad \in \frac{H^3(k, \mathbb{Z}/4\mathbb{Z})}{[A] \cdot H^1(k, \boldsymbol{\mu}_2)}$$

THEOREM 6.1. *The class* $e_3^{16}(A,\sigma)$ *depends only on* (A,σ) (and not on the choice of η).

PROOF. Suppose that $\eta, \eta' \in H^1(k, \mathrm{HSpin}_{16})$ map to $(A,\sigma) \in H^1(k, \mathrm{PSO}_{16})$. We consider the image $\tau(\eta')$ of η' in the twisted group $H^1(k, (\mathrm{HSpin}_{16})_\eta)$. Since $\tau(\eta')$ maps to zero in $H^1(k, (\mathrm{PSO}_{16})_\eta)$, it is the image of some $\zeta \in H^1(k, \boldsymbol{\mu}_2)$, where

$\boldsymbol{\mu}_2$ denotes the center of $(\mathrm{HSpin}_{16})_\eta$. In the diagram

$$\begin{array}{ccccc} H^1(k,\mathrm{HSpin}_{16}) & \longrightarrow & H^1(k,E_8) & \xrightarrow{r_{E_8}} & H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \\ \cong\downarrow\tau & & \cong\downarrow\tau & & \downarrow ?-r_{E_8}(\eta) \\ H^1(k,(\mathrm{HSpin}_{16})_\eta) & \longrightarrow & H^1(k,(E_8)_\eta) & \xrightarrow{r_{(E_8)_\eta}} & H^3(k,\mathbb{Q}/\mathbb{Z}(2)), \end{array} \tag{6.2}$$

the left box obviously commutes and the right box commutes by [**Gi**, p. 76, Lemma 7]. Commutativity of the diagram and 5.3 give that

$$r_{E_8}(\eta') = r_{E_8}(\eta) + \zeta \cdot [A],$$

as desired. □

EXAMPLE 6.3. If A is split, then σ is adjoint to a quadratic form q_σ, and $e_3^{16}(A,\sigma)$ equals the Arason invariant of q_σ. Indeed, if A is split, then there is a class $\gamma \in H^1(k,\mathrm{Spin}_{16})$ that maps to η. Statement (5.2) gives:

$$r_{E_8}(\eta) = r_{\mathrm{Spin}_{16}}(\gamma) = e_3(q_\sigma).$$

REMARK 6.4. *If $e_3^{16}(A,\sigma)$ is zero, then there is a class $\eta' \in H^1(k,\mathrm{HSpin}_{16})$ that maps to (A,σ) in $H^1(k,\mathrm{PSO}_{16})$ and has $r_{E_8}(\eta')=0$.* Indeed, for η as at the beginning of this section, the hypothesis implies that $r_{E_8}(\eta) = \zeta\cdot[A]$ for some $\zeta \in H^1(k,\boldsymbol{\mu}_2)$. The element $\eta' := \zeta\cdot\eta$ has the desired properties by the proof of Theorem 6.1.

Part II. The invariant e_3^{16} on decomposable involutions

The purpose of this part is to compute $e_3(A,\sigma)$ in case (A,σ) can be written as $(Q,{}^-)\otimes(C,\gamma)$ where $(Q,{}^-)$ is a quaternion algebra endowed with its canonical involution and (C,γ) is a central simple algebra of degree 8 with symplectic involution. We do this by computing the value of the Rost invariant of E_8 on a subgroup $\mathrm{PGL}_2\times\mathrm{PSp}_8\times\boldsymbol{\mu}_2$; this finer computation will be used in §11.

7. An inclusion $\mathrm{PGL}_2\times\mathrm{PSp}_8\times\boldsymbol{\mu}_2 \subset \mathrm{HSpin}_{16}$

7.1. INCLUSIONS. We now describe an inclusion $\mathrm{PGL}_2\times\mathrm{PSp}_8 \to \mathrm{HSpin}_{16}$. Write S_n for the n-by-n matrix whose only nonzero entries are 1s on the "second diagonal", i.e., in the $(j,n+1-j)$-entries for various j. We identify Sp_{2n} with the symplectic group of $M_{2n}(k)$ endowed with the involution γ_{2n} defined by

$$\gamma_{2n}(x) = \mathrm{Int}\begin{pmatrix} 0 & S_n \\ -S_n & 0\end{pmatrix}^{-1} x^t.$$

We identify Spin_{2n} with the spin group of $M_{2n}(k)$ endowed with the involution σ_{2n} defined by

$$\sigma_{2n}(x) = \mathrm{Int}(S_{2n})\, x^t.$$

These are the realizations of the groups (stated on the level of Lie algebras) given in [**Bou**, §VIII.13].

We now define isomorphisms

$$(M_2(k),\gamma_2)\otimes(M_8(k),\gamma_8) \xrightarrow{\sim} (M_{16}(k),\sigma'_{16}) \xrightarrow{\sim} (M_{16}(k),\sigma_{16}), \tag{7.2}$$

where

$$\sigma'_{16}(x) = \mathrm{Int}\begin{pmatrix} 0 & 0 & 0 & S_4 \\ 0 & 0 & -S_4 & 0 \\ 0 & -S_4 & 0 & 0 \\ S_4 & 0 & 0 & 0 \end{pmatrix} x^t.$$

We take the first isomorphism to be the usual Kronecker product defined on the standard basis elements by

$$E_{ij} \otimes E_{qr} \mapsto E_{8(i-1)+q,8(j-1)+r}.$$

The second isomorphism is conjugation by the matrix

$$\begin{pmatrix} 1_4 & 0 & 0 & 0 \\ 0 & 0 & 1_4 & 0 \\ 0 & -1_4 & 0 & 0 \\ 0 & 0 & 0 & 1_4 \end{pmatrix}$$

where 1_4 denotes the 4-by-4 identity matrix.

The homomorphism of groups induces a map on coroot lattices (= root lattices for the dual root systems) that describes the restriction of the group homomorphism to Cartan subalgebras on the level of Lie algebras. Using the concrete description of the group homomorphism above and the choice of Cartan, etc., from [**Bou**], we see that the map on coroots is given by Table 7B, where the simple roots of $\mathrm{SL}_2, \mathrm{Sp}_8, \mathrm{Spin}_{16}$, and E_8 are labelled α, γ, δ, and ε respectively and are numbered as in Figures 5A and 7A. For $\mathrm{SL}_2, \mathrm{Spin}_{16}$, and E_8, we fix the metric so that roots have

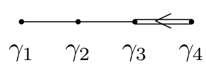


FIGURE 7A. Dynkin diagram of C_4

length 2, which identifies the coroot and root lattices. The inclusion of HSpin_{16} in E_8 is given by (5.1).

		in D_8		in E_8
α_1	$\mapsto$	$\delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4 +$ $+ 3\delta_5 + 2\delta_6 + \delta_7$	$\mapsto$	$-2\varepsilon_1 - 2\varepsilon_2 - 4\varepsilon_3 - 4\varepsilon_4 - 2\varepsilon_5$
$\check\gamma_1$	$\mapsto$	$\delta_1 - \delta_7$	$\mapsto$	$-2\varepsilon_1 - 4\varepsilon_2 - 4\varepsilon_3 - 6\varepsilon_4$ $- 5\varepsilon_5 - 4\varepsilon_6 - 3\varepsilon_7 - 2\varepsilon_8$
$\check\gamma_2$	$\mapsto$	$\delta_2 - \delta_6$	$\mapsto$	$-\varepsilon_4 + \varepsilon_8$
$\check\gamma_3$	$\mapsto$	$\delta_3 - \delta_5$	$\mapsto$	$-\varepsilon_5 + \varepsilon_7$
$\check\gamma_4$	$\mapsto$	$\delta_4 + 2\delta_5 + 2\delta_6 + \delta_7 + \delta_8$	$\mapsto$	$\varepsilon_2 + \varepsilon_3 + 2\varepsilon_4 + 2\varepsilon_5 + \varepsilon_6$

TABLE 7B. Homomorphisms $\mathrm{SL}_2 \times \mathrm{Sp}_8 \to \mathrm{Spin}_{16} \to E_8$ on the level of coroots

Either from the explicit tensor product in (7.2) or from the description of the center of Sp_8 from [**GQ 08**, 8.5], we deduce inclusions:

$$(\mathrm{SL}_2 \times \mathrm{Sp}_8)/\boldsymbol{\mu}_2 \subset \mathrm{Spin}_{16} \quad \text{and} \quad \mathrm{PGL}_2 \times \mathrm{PSp}_8 \subset \mathrm{HSpin}_{16}.$$

Since the short coroot $\check\gamma_4$ of Sp_8 maps to a (co)root in D_8, the homomorphism $\mathrm{Sp}_8 \to \mathrm{Spin}_{16}$ has Rost multiplier 1.

(The statements in the previous paragraph can also be deduced from the branching tables in [**MP**, p. 295], but of course those tables were constructed using data as in Table 7B. To get the statement on Rost multipliers, one uses [**Mer 03**, 7.9].)

We find a subgroup $\mathrm{PGL}_2 \times \mathrm{PSp}_8 \times \boldsymbol{\mu}_2$ of HSpin_{16} by taking the center of HSpin_{16} for the copy of $\boldsymbol{\mu}_2$.

7.3. The composition

$$H^1(k, \mathrm{PGL}_2 \times \mathrm{PSp}_8 \times \boldsymbol{\mu}_2) \to H^1(k, \mathrm{HSpin}_{16}) \to H^1(k, E_8) \xrightarrow{r_{E_8}} H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

defines an invariant of triples $(Q, (C, \gamma), c)$ where Q is a quaternion algebra, (C, γ) is a central simple algebra of degree 8 with symplectic involution, and c is in $k^\times/k^{\times 2}$. We abuse notation and write also r_{E_8} for this invariant.

For example, tracing through the proof of Th. 6.1, we find:

$$r_{E_8}(Q, (C, \gamma), c) = r_{E_8}(Q, (C, \gamma), 1) + (c) \cdot [Q \otimes C]. \tag{7.4}$$

8. Crux computation

LEMMA 8.1. *The composition*

$$H^1(k, \mathrm{PGL}_2) \times 1 \subset H^1(k, \mathrm{PGL}_2) \times H^1(k, \mathrm{PSp}_8 \times \boldsymbol{\mu}_2) \to H^1(k, E_8)$$

is zero.

We warm up by doing a toy version of a computation necessary for the proof of the lemma.

EXAMPLE 8.2. Let O_n be the orthogonal group of the symmetric bilinear form f with matrix S_n as in 7.1. Fix a quadratic extension $k(\sqrt{a})/k$. For ι the nontrivial k-automorphism of $k(\sqrt{a})$ and $c \in k^\times$, the 1-cocycle $\eta \in Z^1(k(\sqrt{a})/k, \mathrm{O}_2)$ defined by

$$\eta_\iota = \begin{pmatrix} 0 & c \\ c^{-1} & 0 \end{pmatrix}$$

defines a bilinear form f_η over k. It is the restriction of f to the k-subspace of $k(\sqrt{a})^2$ of elements fixed by the semilinear automorphism $\eta_\iota \circ \iota$. This subspace has basis

$$\begin{pmatrix} c \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c\sqrt{a} \\ -\sqrt{a} \end{pmatrix},$$

so f_η is isomorphic to $\langle 2c \rangle \langle 1, -a \rangle$.

PROOF OF LEMMA 8.1. Fix a cocycle $\eta \in Z^1(k, \mathrm{PGL}_2)$. The rank 4 maximal torus in PSp_8 (intersection with the torus in E_8 specified by the pinning) is centralized by the image of η, so it gives a k-split torus S in the twisted group $(E_8)_\eta$. A semisimple anisotropic kernel of $(E_8)_\eta$ is contained in the derived subgroup D of the centralizer of S. The root system of D (over an algebraic closure of k) consists of the roots of E_8 orthogonal to the elements of the coroot lattice with image lying in S, which are given in Table 7B. The roots of D form a system of rank 4 with simple roots $\phi_1, \phi_2, \phi_3, \phi_4$ as in Table 8A; they span a system of type D_4 with Dynkin diagram as in Figure 8B, where $\tilde{\phi}$ is the highest root $\phi_1 + 2\phi_2 + \phi_3 + \phi_4 = \tilde{\varepsilon}$.

D_4	ϕ_1	ϕ_2	ϕ_3	ϕ_4
E_8	ε_6	$\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4 + \varepsilon_5$	$\varepsilon_5 + \varepsilon_6 + \varepsilon_7$	$\varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8$

TABLE 8A. Simple roots in the centralizer of the C_4-torus in E_8

We now compute the map $\mathrm{SL}_2 \to D$. On the level of tori, it is given by Table 7B. On the level of Lie algebras, we compute using the explicit map (7.2) that the element $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ of $\mathfrak{sl}_2$ maps to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \left(\begin{smallmatrix} 0 & 0 & 1_4 & 0 \\ 0 & 0 & 0 & 1_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & 1_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_4 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

in $M_{16}(k)$. In terms of the Chevalley basis of the Lie algebra of E_8, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ maps to

$$X_{\phi_1} + X_{\phi_3} + X_{\phi_4} + X_{-\tilde{\phi}}.$$

The cocycle η represents a quaternion algebra (a, b) over k. If it is split, then η is zero and we are done. Otherwise, a is not a square. Replacing η with an equivalent cocycle, we may assume that η belongs to $Z^1(k(\sqrt{a})/k, \mathrm{PGL}_2)$ and takes the value

$$\eta_\iota = \mathrm{Int}\left(\begin{smallmatrix} 0 & b \\ 1 & 0 \end{smallmatrix}\right) = \mathrm{Int}\left(\begin{smallmatrix} 0 & -\sqrt{-b} \\ 1/\sqrt{-b} & 0 \end{smallmatrix}\right)$$

on the non-identity k-automorphism ι of $k(\sqrt{a})$. That is, η_ι is conjugation by the element $w_{\alpha_1}(-\sqrt{-b})$ in Steinberg's notation for generators of a Chevalley group from [**St**]. The image of η in E_8 is the cocycle $\hat{\eta}$ with

$$\hat{\eta}_\iota := \prod_\phi w_\phi(-\sqrt{-b}) \tag{8.3}$$

where ϕ ranges over the set $\Sigma := \{-\tilde{\phi}, \phi_1, \phi_3, \phi_4\}$. (The order of terms in the product does not matter, as the roots in Σ are pairwise strongly orthogonal.) We compute the action of this element on each $x_\phi : \mathbb{G}_a \to D_4$ for $\phi \in \Sigma$. Using the orthogonality of the roots in Σ, we have:

$$\mathrm{Int}(\hat{\eta}_\iota) x_\phi(u) = \mathrm{Int}(w_\phi(-\sqrt{-b})) x_\phi(u) = x_\phi(u/b), \tag{8.4}$$

where the second equality is by the identities in [**St**, p. 66].

We now identify D (over an algebraic closure) with Spin_8 using the pinning of Spin_8 from [**Bou**] and project $\hat{\eta}$ to a 1-cocycle with values in SO_8. This cocycle defines a quadratic form q that we claim is hyperbolic. Indeed, from equation (8.4), we deduce that the image of $\hat{\eta}$ in SO_8 is the matrix

$$\begin{pmatrix} &&&&&&& -1 \\ &&&&&& 1/b & \\ &&&&& -b && \\ &&&& 1 &&& \\ &&& 1 &&&& \\ && -1/b &&&&& \\ & b &&&&&& \\ -1 &&&&&&& \end{pmatrix}$$

This preserves the hyperbolic planes in k^8 spanned by the 1st and 8th, 2nd and 7th, etc., standard basis vectors, so we can compute the quadratic form by restricting to each of these planes as in Example 8.2. One finds that q is isomorphic to

$$\langle 2 \rangle \otimes \left\langle -1, b^{-1}, -b, 1 \right\rangle \otimes \langle 1, -a \rangle,$$

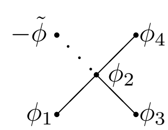


FIGURE 8B. Extended Dynkin diagram of centralizer of the C_4-torus in E_8

which is hyperbolic because the middle term is. In particular, the twisted group $(\mathrm{SO}_8)_{\hat\eta}$ is split, and the same is true for D. We conclude that $(E_8)_\eta$ is split and the image of η in $H^1(k, E_8)$ is zero. □

REMARK 8.5. In the case where k contains a square root of every element of the prime field F, one can give an easier proof of Lemma 8.1 as follows. Repeat the first paragraph. The composition

$$H^1(*, \mathrm{PGL}_2) \to H^1(*, E_8) \xrightarrow{r_{E_8}} H^3(*, \mathbb{Q}/\mathbb{Z}(2))$$

gives a normalized invariant of PGL_2, which is necessarily of the form $Q \mapsto [Q]\cdot x$ for some fixed $x \in H^1(F, \mathbb{Z}/2\mathbb{Z})$ by [**Se 03**, 18.1, §23]. Thus every element of $H^1(k, E_8)$ coming from $H^1(k, \mathrm{PGL}_2)$ has Rost invariant zero (because x is killed by k) and is isotropic (obviously), hence is zero by Prop. 12.1(1) below.

9. Rost invariant

THEOREM 9.1. *The composition*

$$H^1(k, \mathrm{PGL}_2) \times H^1(k, \mathrm{PSp}_8) \times H^1(k, \boldsymbol{\mu}_2) \to H^1(k, E_8) \xrightarrow{r_{E_8}} H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

is given by

$$(Q, (C, \gamma), c) \mapsto \Delta(C, \gamma) + (c) \cdot [Q \otimes C] \quad \in H^3(k, \mathbb{Z}/2\mathbb{Z}).$$

Here Δ refers to the discriminant of symplectic involutions on algebras of degree 8 defined in [**GPT**].

PROOF. By (7.4), it suffices to prove the case where $c = 1$.

Step 1. We first verify the proposition in case C has index at most 2 and γ is hyperbolic; we write $(C, \gamma) = (Q' \otimes M_4(k), ^- \otimes \mathrm{hyp})$ for some quaternion algebra Q'. In this way, we restrict r_{E_8} to an invariant $H^1(k, \mathrm{PGL}_2) \times H^1(k, \mathrm{PGL}_2) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$, which we claim is zero.

We argue as in [**Se 03**, §17]. We view $H^1(k, \mathrm{PGL}_2)$ as the image of $H^1(k, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ via the map that sends elements $a, b \in k^\times/k^{\times 2}$ to the quaternion algebra (a, b). By restriction, r_{E_8} can be viewed as an invariant of $(\mathbb{Z}/2\mathbb{Z})^{\times 4}$; its image consists of elements killed by a quadratic extension (by Lemma 8.1), so belong to the 2-torsion in $H^3(k, \mathbb{Z}/4\mathbb{Z})$, i.e., $H^3(k, \mathbb{Z}/2\mathbb{Z})$. Because the image of $H^1(k, (\mathbb{Z}/2\mathbb{Z})^{\times 4})$ lies in $H^3(k, \mathbb{Z}/2\mathbb{Z})$ and the value of r_{E_8} on an element is unaltered if we interchange the first two coordinates (corresponding to the quaternion algebra Q) or the third and fourth coordinates (corresponding to Q'), we deduce that r_{E_8} is of the form

$$Q, Q' \mapsto \lambda_0 + \lambda_Q \cdot [Q] + \lambda_{Q'} \cdot [Q']$$

for uniquely determined elements $\lambda_0, \lambda_Q, \lambda_{Q'}$ in $H^\bullet(k, \mathbb{Z}/2\mathbb{Z})$. (There is no term involving $[Q] \cdot [Q']$ because such a term would have degree at least 4.)

The element λ_0 is zero, because the Rost invariant r_{E_8} is normalized. The coefficient λ_Q is zero by Lemma 8.1.

Write K for the field obtained by adjoining indeterminates a, b to k and Q^{gen} for the generic quaternion algebra (a, b) over K. On the one hand, the value of r_{E_8} on the triple

$$(Q^{\mathrm{gen}}, (Q^{\mathrm{gen}} \otimes M_4(k), ^- \otimes \mathrm{hyp}), 1) \quad \text{is} \quad \lambda_{Q'} \cdot [Q^{\mathrm{gen}}].$$

On the other hand, the algebra with involution

$$(A, \sigma) := (Q^{\mathrm{gen}}, ^-) \otimes (Q^{\mathrm{gen}} \otimes M_4(k), ^- \otimes \mathrm{hyp})$$

has A split and σ hyperbolic, so $e_3^{16}(A,\sigma)$ is zero as an element of $H^3(k,\mathbb{Z}/4\mathbb{Z})$. Thus $\lambda_{Q'}=0$ and r_{E_8} sends Q,Q' to 0. This verifies the proposition for C of index at most 2 and γ hyperbolic.

Step 2. For a fixed quaternion algebra Q, the map

$$(C,\gamma)\mapsto r_{E_8}(Q,(C,\gamma),1) \quad \in H^3(K,\mathbb{Z}/2\mathbb{Z})$$

defines an invariant of $H^1(k,\mathrm{PSp}_8)$ that is zero on the trivial class by Lemma 8.1, hence by [**GPT**, 4.1] is of the form

$$(C,\gamma)\mapsto \lambda_1\cdot[C]+\lambda_0\cdot\Delta(C,\gamma) \tag{9.2}$$

for uniquely determined elements $\lambda_i\in H^i(k,\mathbb{Z}/2\mathbb{Z})$ for $i=0,1$ (which may depend on Q).

In case C has index 2 and γ is hyperbolic, (C,γ) maps to zero by Step 1 and $\Delta(C,\gamma)=0$, so $\lambda_1=0$.

We are left with deciding whether the element λ_0 is 0 or 1 in $H^0(k,\mathbb{Z}/2\mathbb{Z})$. Suppose that C has index 2 and put η for the image of $(Q,(C,\mathrm{hyp}),1)$ in $H^1(k,E_8)$. The homomorphism

$$\mathrm{Sp}(C,\mathrm{hyp})=(\mathrm{Sp}_8)_\eta\to(E_8)_\eta$$

has Rost multiplier 1 by 7.1, i.e., the induced map

$$H^1(k,\mathrm{Sp}(C,\mathrm{hyp}))\to H^1(k,(E_8)_\eta)\xrightarrow{r_{E_8}} H^3(k,\mathbb{Q}/\mathbb{Z}(2))$$

is the Rost invariant of $\mathrm{Sp}(C,\mathrm{hyp})$. In particular it is not zero, so λ_0 is not zero, i.e., $\lambda_0=1$. This proves the proposition. □

10. Comparison of e_3^{16} and e_3^{hyp}

Let Q be a quaternion algebra and let (C,γ) be a central simple algebra of degree 8 with symplectic involution. The tensor product $(Q,{}^-)\otimes(C,\gamma)$ has degree 16 (clearly), trivial discriminant [**KMRT**, 7.3(5)], and one component of the even Clifford algebra is split [**Tao**, 4.15, 4.16], so the tensor product belongs to I^3.

Corollary 10.1 (of Th. 9.1). *We have:*

$$e_3^{16}\,[(Q,{}^-)\otimes(C,\gamma)]=\Delta(C,\gamma)\quad\in H^3(k,\mathbb{Z}/4\mathbb{Z})/[Q\otimes C]\cdot H^1(k,\boldsymbol{\mu}_2).\quad\square$$

We compare the invariants e_3^{16} (from §6) and e_3^{hyp} (from §3). The invariant e_3^{16} is only defined on (A,σ) in I^3 where A has degree 16. For such (A,σ), the invariant e_3^{hyp} is only defined if A is isomorphic to $M_2(C)$ for a central simple algebra C of degree 8. We show:

Corollary 10.2. *If (A,σ) is in I^3 and A is isomorphic to $M_2(C)$, then*

$$e_3^{16}(A,\sigma)=e_3^{\mathrm{hyp}}(A,\sigma)+\Delta(C)\quad\in H^3(k,\mathbb{Z}/4\mathbb{Z})/[C]\cdot H^1(k,\boldsymbol{\mu}_2).$$

Recall the definition of $\Delta(C)$ from [**GPT**, §11]: For a symplectic involution γ on C, the image of the discriminant $\Delta(C,\gamma)$ in $H^3(k,\mathbb{Z}/2\mathbb{Z})/[C]\cdot H^1(k,\boldsymbol{\mu}_2)$ depends only on C and not on γ, and we write $\Delta(C)$ for that image. If C is decomposable—i.e., isomorphic to a tensor product of three quaternion algebras—then $\Delta(C)$ is zero. But for indecomposable C, it is not known whether $\Delta(C)$ is zero [**GPT**, 11.2].

Proof of Cor. 10.2. Write $^-$ for the canonical symplectic involution on $M_2(k)$; it is hyperbolic. Fix a symplectic involution γ on C. We fix also an isomorphism of $M_2(k)\otimes C$ with A; translating the involution $^-\otimes\gamma$ to A gives a hyperbolic involution hyp. By Cor. 10.1, we have:

$$e_3^{16}(A, \mathrm{hyp}) = \Delta(C) \quad \in H^3(k, \mathbb{Z}/4\mathbb{Z})/[C]\cdot H^1(k, \boldsymbol{\mu}_2).$$

Let $\eta, \eta' \in H^1(k, \mathrm{HSpin}_{16})$ have image $(A, \mathrm{hyp}), (A, \sigma)$ in $H^1(k, \mathrm{PSO}_{16})$ respectively. The bottom row of (6.2) can be rewritten as

$$H^1(k, \mathrm{HSpin}(A, \mathrm{hyp})) \to H^1(k, (E_8)_\eta) \xrightarrow{r_{(E_8)_\eta}} H^3(k, \mathbb{Q}/\mathbb{Z}(2)).$$

For $\nu \in H^1(k, \mathrm{Spin}(A, \mathrm{hyp}))$ mapping to $\tau(\eta') \in H^1(k, \mathrm{HSpin}(A, \mathrm{hyp}))$, we have:

$$r_{(E_8)_\eta}(\tau(\eta')) = r_{\mathrm{Spin}(A,\mathrm{hyp})}(\nu) \quad \text{by (5.2)}.$$

The equality

$$e_3^{\mathrm{hyp}}(A,\sigma) = e_3^{16}(A,\sigma) - e_3^{16}(A, \mathrm{hyp}),$$

follows by commutativity of (6.2), which completes the proof of the corollary. □

As $\Delta(C)$ vanishes on decomposable algebras C, we immediately obtain:

Corollary 10.3. *If (A,σ) is in I^3 and A is isomorphic to $M_2(Q_1\otimes Q_2\otimes Q_3)$ for some quaternion algebras Q_1, Q_2, Q_3 (e.g., this holds if A has index ≤ 4), then*

$$e_3^{16}(A,\sigma) = e_3^{\mathrm{hyp}}(A,\sigma) \quad \in H^3(k, \mathbb{Z}/4\mathbb{Z})/[A]\cdot H^1(k, \boldsymbol{\mu}_2). \quad \square$$

Part III. Groups of type E_8 constructed from 9 parameters

11. Construction of E_8's

In 7.1, we gave a concrete description of an inclusion of $\mathrm{PGL}_2 \times \mathrm{PSp}_8 \times \boldsymbol{\mu}_2$ in HSpin_{16} which in turn includes in E_8. Similarly, we can give an explicit embedding of $\mathrm{PGL}_2^{\times 3}$ in PSp_8 as in [**Dy**, Table 9]. On the level of coroot lattices the total inclusion

(11.1) $$\mathrm{PGL}_2^{\times 4} \times \boldsymbol{\mu}_2 \subset \mathrm{PGL}_2 \times \mathrm{PSp}_8 \times \boldsymbol{\mu}_2 \subset \mathrm{HSpin}_{16} \subset E_8$$

is described in Table 11. We remark that the four copies of PGL_2 are not normalized by a maximal torus of E_8.

simple (co)root in copy of PGL_2	in C_4	in E_8
α_1		$-(2\varepsilon_1 + 2\varepsilon_2 + 4\varepsilon_3 + 4\varepsilon_4 + 2\varepsilon_5)$
α_2	$\check\gamma_1 - \check\gamma_3$	$-(2\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 + 6\varepsilon_4 + 4\varepsilon_5 + 4\varepsilon_6 + 4\varepsilon_7 + 2\varepsilon_8)$
α_3	$\check\gamma_1 + \check\gamma_3$	$-(2\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 + 6\varepsilon_4 + 6\varepsilon_5 + 4\varepsilon_6 + 2\varepsilon_7 + 2\varepsilon_8)$
α_4	$\check\gamma_1 + 2\check\gamma_2 + \check\gamma_3$	$-(2\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 + 8\varepsilon_4 + 6\varepsilon_5 + 4\varepsilon_6 + 2\varepsilon_7)$

Table 11. Inclusion $\mathrm{PGL}_2^{\times 4} \subset \mathrm{PGL}_2 \times \mathrm{PSp}_8 \subset E_8$ on the level of coroots. α_i is a simple (co)root in the i-th copy of PGL_2. That copy is inside PSp_8 for $i \neq 1$.

Applying Galois cohomology to (11.1) gives a function

(11.2) $$H^1(k, \mathrm{PGL}_2^{\times 4} \times \boldsymbol{\mu}_2) \to H^1(k, E_8).$$

The first set classifies quadruples (Q_1, Q_2, Q_3, Q_4) of quaternion k-algebras together with an element $c \in k^\times/k^{\times 2}$, and the second set classifies groups of type E_8 over k. Therefore we may view the function (11.2) as a construction of groups of type E_8 via Galois descent.

Corollary 11.3 (of Theorem 9.1). *The Rost invariant of a group of type* E_8 *constructed from* (Q_1, Q_2, Q_3, Q_4, c) *is* $(c) \cdot \sum [Q_i]$.

Proof. As (C, γ) is the tensor product $\otimes_{i=2}^4 (Q_i, ^-)$, it is decomposable, and so has discriminant zero [**GPT**]. Theorem 9.1 gives the claim. ☐

11.4. How much can we vary the data (Q_1, Q_2, Q_3, Q_4, c) without changing the resulting group of type E_8? For example, let Q'_2, Q'_3, Q'_4 be quaternion algebras such that the tensor products $\otimes_{i=2}^4 (Q'_i, ^-)$ and $\otimes_{i=2}^4 (Q_i, ^-)$ are isomorphic as algebras with involution. Then the images of (Q_1, Q_2, Q_3, Q_4, c) and $(Q_1, Q'_2, Q'_3, Q'_4, c)$ in $H^1(k, \mathrm{PGL}_2) \times H^1(k, \mathrm{PSp}_8) \times H^1(k, \boldsymbol{\mu}_2)$ agree, hence one obtains the same group of type E_8 from the two inputs.

We also have:

Proposition 11.5. *For every permutation* π*, the group of type* E_8 *constructed from* $(Q_{\pi 1}, Q_{\pi 2}, Q_{\pi 3}, Q_{\pi 4}, c)$ *is the same.*

Proof. We compare the images η and η_π of tuples (Q_1, Q_2, Q_3, Q_4, c) and $(Q_{\pi 1}, Q_{\pi 2}, Q_{\pi 3}, Q_{\pi 4}, c)$ respectively in $H^1(k, \mathrm{HSpin}_{16})$. As both η and η_π map to the class of $\otimes (Q_i, ^-)$ in $H^1(k, \mathrm{PSO}_{16})$, the class of η_π is $\zeta_\pi \cdot \eta$ for some $\zeta_\pi \in H^1(k, \boldsymbol{\mu}_2)$. The element ζ_π is uniquely determined as an element of the abelian group $\Gamma := H^1(k, \boldsymbol{\mu}_2)/\operatorname{im}(\mathrm{PSO}_{16})_\eta(k)$, where $(\mathrm{PSO}_{16})_\eta(k)$ maps into $H^1(k, \boldsymbol{\mu}_2)$ via the connecting homomorphism arising from the exact sequence at the bottom of diagram (4.2), see [**Se 02**, §I.5.5, Cor. 2]. This defines a homomorphism ζ from the symmetric group on 4 letters, $\mathcal{S}_4$, to Γ.

As Γ is abelian, the homomorphism ζ factors through the commutator subgroup of $\mathcal{S}_4$, the alternating group. But ζ vanishes on the odd permutation $(3\,4)$ by 11.4, so ζ is the zero homomorphism. This proves the proposition. ☐

11.6. Tits's construction. In [**Ti 66a**], Tits gave a construction of algebraic groups of type E_8 with inputs an octonion algebra and an Albert algebra. In terms of algebraic groups, there is an (essentially unique) inclusion of $G_2 \times F_4$ in E_8 [**Dy**, p. 226], and Tits's construction is the resulting map in Galois cohomology:

$$H^1(k, G_2) \times H^1(k, F_4) \to H^1(k, E_8).$$

His construction and ours from (11.2) overlap, but they are distinct.

We compute the Rost invariant of a group G of type E_8 constructed by Tits's recipe from an octonion algebra with 3-Pfister norm form γ_3 and an Albert algebra A. Because the inclusions $G_2 \subset E_8$ and $F_4 \subset E_8$ both have Rost multiplier 1 [**Dy**, p. 192], we have:

$$r_{E_8}(G) = e_3(\gamma_3) + r_{F_4}(A)$$

by the argument in the proof of [**GQ 08**, Lemma 5.6], where r_{F_4} denotes the Rost invariant relative to the split group of type F_4. Associated with A are Pfister forms ϕ_3 and ϕ_5, where ϕ_i has dimension 2^i and ϕ_3 divides ϕ_5, see [**Se 03**, 22.5]. We find:

$$15 r_{E_8}(G) = e_3(\gamma_3 + \phi_3) \quad \in H^3(k, \mathbb{Z}/2\mathbb{Z}).$$

12. Tits index of groups of type E_8

In this section, we note some relationships between the Tits index of a group G of type E_8 over k and its Rost invariant $r_{E_8}(G)$.

Recall that if $r_{E_8}(G)$ is killed by a quadratic extension or is 2-torsion, then it belongs to $H^3(k, \mathbb{Z}/2\mathbb{Z})$. A *symbol* is an element of the image of the cup product map $H^1(k, \mathbb{Z}/2\mathbb{Z})^{\times 3} \to H^3(k, \mathbb{Z}/2\mathbb{Z})$. The *symbol length* of an element $x \in H^3(k, \mathbb{Z}/2\mathbb{Z})$ is the smallest integer n such that x is equal to a sum of n symbols in $H^3(k, \mathbb{Z}/2\mathbb{Z})$. Zero is the unique element with symbol length 0.

PROPOSITION 12.1. *Let G be an isotropic group of type E_8. Then:*

(1) *If $r_{E_8}(G)$ is zero, then G is split.*
(2) *If $r_{E_8}(G)$ is split by a quadratic extension of k, then $r_{E_8}(G)$ has symbol length ≤ 3 in $H^3(k, \mathbb{Z}/2\mathbb{Z})$.*
(3) *If $r_{E_8}(G)$ is 2-torsion and G has k-rank ≥ 2, then the Tits index is given by Table 12.*

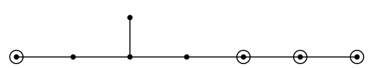
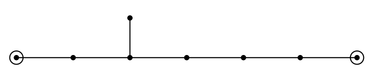
index	symbol length of $r_{E_8}(G)$
split	0
	1
	2

TABLE 12. Tits index versus symbol length for isotropic groups of type E_8 such that $r_{E_8}(G)$ is 2-torsion and G has k-rank ≥ 2

12.2. Before proving the proposition, we give some context for it. We consider a group G constructed via (11.2) from quaternion algebras Q_1, Q_2, Q_3, Q_4 and some $c \in k^\times/k^{\times 2}$. If at least one of the Q_i is split, then G contains a subgroup isomorphic to PGL_2 and so is isotropic. If at least two of the Q_i are split or a tensor product of some three of them is split, then G contains a subgroup isomorphic to $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ or PSp_8 respectively, and so has k-rank ≥ 2. In any case, the Rost invariant of G is zero over $k(\sqrt{c})$, which is either k or a quadratic extension of k.

In particular, if one wishes to use (11.2) to construct groups of type E_8 that are non-split but in the kernel of the Rost invariant, then none of the Q_i can be split, nor can any tensor product of three of them.

EXAMPLE 12.3. If the quaternion algebras Q_2, Q_3, Q_4 are split and $(c)\cdot[Q_1] \neq 0$, then $(c) \cdot [Q_1]$ is a symbol in $H^3(k, \mathbb{Z}/2\mathbb{Z})$ corresponding to a 3-Pfister form q. By Proposition 12.1(3), G has semisimple anisotropic kernel $\mathrm{Spin}(q)$.

EXAMPLE 12.4. For "generic" c and quaternion algebras Q_i, construction (11.2) gives a group G of type E_8 whose Rost invariant is killed by a quadratic extension of k and has symbol length 4. The group G is anisotropic by Prop. 12.1(2).

We prepare the proof of Prop. 12.1 with lemmas on groups of type D_6 and E_7.

Lemma 12.5. *The Witt index of a 12-dimensional quadratic form $q \in I^3$ is given by the table:*

Witt index of q	0	2	6
symbol length of $e_3(q)$ in $H^3(k, \mathbb{Z}/2\mathbb{Z})$	2	1	0

Proof. The Witt index of q cannot be 1 because 10-dimensional forms in I^3 are isotropic. Similarly, it cannot be 3, 4, or 5 by the Arason-Pfister Hauptsatz. This shows that 0, 2, and 6 are the only possibilities.

The form is hyperbolic if and only if it belongs to I^4, i.e., $e_3(q)$ is zero; this proves the last column of the table. If the Witt index is 2, then $q = \langle c \rangle \gamma \oplus 2\mathcal{H}$ for some $c \in k^\times$ and anisotropic 3-Pfister form γ, so $e_3(q)$ is a symbol. Finally, suppose that $e_3(q)$ has symbol length 1, i.e., $q - \gamma$ is in I^4 for some anisotropic 3-Pfister form γ. Over the function field K of γ, the form q is hyperbolic by Arason-Pfister, so $q = \langle c \rangle \gamma \oplus 2\mathcal{H}$ for some $c \in k^\times$ by [**Lam**, X.4.11]. $\square$

In the next lemma, we write E_7 for the split simply connected group of that type, and E_6^K for the quasi-split simply connected group of type E_6 associated with a quadratic étale k-algebra K.

Lemma 12.6. *There is an inclusion of E_6^K in E_7 with Rost multiplier* 1 *such that the induced map $H^1(K/k, E_6^K) \to H^1(K/k, E_7)$ is surjective.*

A class $\eta \in H^1(k, E_7)$ is split by K if and only if K kills the Rost invariant $r_{E_7}(\eta)$ by [**Ga 01b**]. It follows from [**Ga 01b**, 3.6] that there is some quadratic étale k-algebra L such that η is in the image of $H^1(K/k, E_6^L) \to H^1(K/k, E_7)$. The point of the lemma is to arrange that $L = K$.

Proof of Lemma 12.6. We view E_7 as the identity component of the group preserving a quartic form on the 56-dimensional vector space $\left(\begin{smallmatrix} k & J \\ J & k \end{smallmatrix}\right)$ for J the split Albert algebra, cf. [**Ga 01c**]. Write S for the subgroup of E_7 that stabilizes the subspaces $\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)$; it is reductive and its derived subgroup is the split simply connected group of type E_6.

As in [**Ga 01b**, 3.5], for i a square root of -1, the map

$$\left(\begin{smallmatrix} \alpha & x \\ y & \beta \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} i\beta & iy \\ ix & i\alpha \end{smallmatrix}\right)$$

is a $k(i)$-point of E_7, and this gives an inclusion $\boldsymbol{\mu}_4 \hookrightarrow E_7$. Twisting E_7 by a 1-cocycle $\nu \in Z^1(k, \boldsymbol{\mu}_4)$ that maps to the class of K in $H^1(k, \boldsymbol{\mu}_2)$, we find inclusions

$$E_6^K = (E_6)_\nu \subset S_\nu \subset (E_7)_\nu \cong E_7.$$

Write ι for the nontrivial k-automorphism of K. The group S_ν is the intersection $P \cap \iota(P)$ for a maximal parabolic K-subgroup P of $(E_7)_\nu$, hence the map $H^1(K/k, S_\nu) \to H^1(K/k, E_7)$ is surjective [**PR**, pp. 369, 383].

We have an exact sequence

$$1 \longrightarrow E_6^K \longrightarrow S_\nu \xrightarrow{\pi} R^1_{K/k}(\mathbb{G}_m) \longrightarrow 1$$

where π is the map that sends the endomorphism

$$\left(\begin{smallmatrix} \alpha & x \\ y & \beta \end{smallmatrix}\right) \mapsto \begin{pmatrix} \mu\alpha & f(x) \\ f^\dagger(y) & \mu^{-1}\beta \end{pmatrix}$$

to μ. Formula [**Ga 01c**, 1.6] gives an explicit splitting s of π defined over k such the image of s is contained in the parabolic subgroup Q of $(E_6)_\nu$ stabilizing the subspace W from [**Ga 01c**, 6.8].

Fix an element $\eta' \in H^1(K/k, S_\nu)$ that maps to $\eta \in H^1(K/k, E_7)$. We twist S_ν by $s\pi(\eta')$. The image of η'' of η' in $H^1(K/k, (S_\nu)_{s\pi(\eta')})$ under the twisting map takes values in the semisimple part $D := ((E_6)_\nu)_{s\pi(\eta')}$. But D contains the k-parabolic Q, hence D is quasi-split or has semisimple anisotropic kernel a transfer $R_{K/k}(H)$ where H is anisotropic of type ${}^1A_2$. But this is impossible because D is split by a quartic extension of k, so D is the quasi-split group E_6^K. □

Proof of Prop. 12.1. Statement (1) is standard, so we only sketch the proof. The semisimple anisotropic kernel of an isotropic but non-split group of type E_8 is a simply connected group of type E_7, D_7, E_6, D_5, or D_4 by the classification of possible indexes in [**Ti 66b**, p. 60], but the Rost invariant has zero kernel for a split group of any of those types [**Ga 01b**]. Statement (1) now follows by Tits's Witt-type theorem [**Ti 66b**, 2.7.2(d)].

For (3), we may assume that G is not split, equivalently that $r_{E_8}(G)$ has positive symbol length. Because the k-rank of G is at least 2, Tits's table in [**Ti 66b**, p. 60] shows that the semisimple anisotropic kernel of G is a strongly inner simply connected group of type E_6, D_6, or D_4. The first case is impossible because $r_{E_8}(G)$ is 2-torsion. Statement (3) now follows from Tits's Witt-type theorem and Lemma 12.5.

To prove (2), by (3) we may assume that G has k-rank 1, hence that the semisimple anisotropic kernel of G is a strongly inner simply connected group of type D_7 or E_7. In the first case, $r_{E_8}(G)$ is the Arason invariant of a 14-dimensional form in I^3, hence has symbol length ≤ 3 by [**HT**, Prop. 2.3]. For the second case, by Lemma 12.6 it suffices to prove that the Rost invariant of every element of $H^1(K/k, E_6^K)$ has symbol length at most 3, which is [**C**, p. 321]. □

Presumably the methods of [**C**] can be used to give an alternative proof of Prop. 12.1(2) that avoids Lemma 12.6.

13. Reduced Killing form up to Witt-equivalence

Recall that the *reduced Killing form* of an absolutely almost simple algebraic group G — which we denote by $\mathrm{redKill}_G$ — is equal to the usual Killing form divided by twice the dual Coxeter number [**GN**, §5]. For a group G of type E_8, all the roots of G have the same length and the dual Coxeter number equals the (usual) Coxeter number, which is 30. Hence the usual Killing form Kill_G satisfies $\mathrm{Kill}_G = 60\,\mathrm{redKill}_G$ and Kill_G is zero in characteristics 2, 3, 5.

We identify the bilinear form $\mathrm{redKill}_G$ with the quadratic form (and element of the Witt ring) $x \mapsto \mathrm{redKill}_G(x, x)$.

Example 13.1 (the split group). The reduced Killing form for the split group E_8 of that type is Witt-equivalent to the 8-dimensional form $\langle 1, 1, \dots, 1\rangle$, which we denote simply by 8. To see this, note that the positive root subalgebras span a totally isotropic subspace parallel to the isotropic subspace spanned by the negative root subalgebras, so $\mathrm{redKill}_{E_8}$ is Witt-equivalent to its restriction to the Lie algebra of a split maximal torus. By [**GN**], this restriction is isomorphic to the quadratic form $x \mapsto xCx^t$ for C the Cartan matrix of the root system, and it is easy to check that this quadratic form is 8.

EXAMPLE 13.2 (Tits's groups). For a group G of type E_8 obtained from Tits's construction as in 11.6, we have

$$\mathrm{redKill}_G = \langle 2 \rangle \, [8 - (4\gamma_3 + 4\phi_3 + \langle 2 \rangle \gamma_3(\phi_5 - \phi_3))]$$

by [**J**, p. 117, (144)], where the Killing forms for the subalgebras of type F_4 and G_2 are given in [**Se 03**, 27.20].

13.3. The map $G \mapsto \langle 2 \rangle (\mathrm{redKill}_{E_8} - \mathrm{redKill}_G)$ defines a Witt-invariant of $H^1(*, E_8)$ in the sense of [**Se 03**, §27], i.e., a collection of maps

$$\kappa\colon \boxed{\text{groups of type } E_8 \text{ over } K} \to W(K)$$

for every extension K/k (together with some compatibility condition), where $W(K)$ denotes the Witt ring of K.

EXAMPLE 13.4 (groups over $\mathbb{R}$). For each of the three groups of type E_8 over the real numbers, we list the Rost invariant, the (signature of the) Killing form, and the value of κ. All three groups are obtained by Tits's construction [**J**, p, 121], so the Killing form and Rost invariant are provided by the formulas in 13.2 and 11.6. The Rost invariant r_{E_8} takes values in $H^3(\mathbb{R}, \mathbb{Q}/\mathbb{Z}(2)) = H^3(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

G	$r_{E_8}(G)$	Kill_G	$\kappa(G)$
split	0	8	0
other	1	-24	$32 \in I^5$
anisotropic/compact	0	-248	$256 \in I^8$

The rest of this section is concerned with the question over a field k: *In what power of I does $\kappa(G)$ lie?*

LEMMA 13.5. *For every group G of type E_8, $\kappa(G)$ belongs to I^3.*

The proof is not at all special to E_8. It can be used, e.g., to give an alternative proof of [**BMT**, Th. 4] that avoids referring to [**KMRT**, 31.20].

PROOF OF LEMMA 13.5. Because E_8 is simply connected, the adjoint representation $E_8 \to \mathrm{O}(\mathrm{redKill}_{E_8})$ lifts to a homomorphism $E_8 \to \mathrm{Spin}(\mathrm{redKill}_{E_8})$. For G a group of type E_8 over an extension K/k, the reduced Killing form of G is the image of G under the composition

$$H^1(K, E_8) \to H^1(K, \mathrm{Spin}(\mathrm{redKill}_{E_8})) \to H^1(K, \mathrm{O}(\mathrm{redKill}_{E_8})),$$

so it belongs to I^3K, see e.g. [**KMRT**, p. 437]. □

13.6. HISTORICAL NOTE. If k is a number field, then $\kappa(G)$ is in $I^5(k)$ for every k-group G of type E_8. To see this, note that $\kappa(G)$ is hyperbolic at finite places (because $\kappa(G)$ is in I^3) and at real places is 0, 32, or 256 by Example 13.4. Then you can use strong approximation to construct a form in $I^5(k)$ that agrees with $\kappa(G)$ at all real places, hence equals $\kappa(G)$ by the Hasse-Minkowski Principle.

Everything that went into this observation was published by 1971, if not a long time before. This led to the question: *Does $\kappa(G)$ belong to $I^4(k)$ or even $I^5(k)$ for every group G of type E_8 over every field k (of characteristic $\neq 2$)?* The appearance of the Rost invariant settled it negatively, as we now show.

LEMMA 13.7. *For every group G of type E_8, we have $30r_{E_8}(G) = e_3(\kappa(G))$. The form $\kappa(G)$ is in I^4 if and only if $30r_{E_8}(G) = 0$.*

PROOF. We have a commutative diagram, where $\mathfrak{e}_8$ denotes the Lie algebra of E_8 and arrows are labeled with Rost multipliers:

$$\begin{array}{ccc} E_8 & \longrightarrow & \mathrm{Spin}(\mathfrak{e}_8) \\ & \searrow^{60} & \downarrow{\scriptstyle 2} \\ & & \mathrm{SL}(\mathfrak{e}_8) \end{array}$$

Therefore, the Rost multiplier of the top arrow is $60/2 = 30$ [**Mer 03**, p. 122]. The first claim follows. The second claim amounts to (2.2): the kernel of e_3 is I^4. $\square$

EXAMPLE 13.8. For every field k, there is a versal E_8-torsor G defined over some extension K/k [**Se 03**, 5.3]. As the Rost invariant r_{E_8} has order 60 [**Mer 03**, 16.8], it follows that $r_{E_8}(G)$ has order 60 in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ [**Se 03**, 12.3]. Conflating E_8-torsors with groups of type E_8, we obtain: *$\kappa(G)$ is in $I^3(K)$ but not $I^4(K)$.*

In the same spirit as the question from 13.6 and in view of what we have just seen, one might ask:

QUESTION 13.9. *If G is a group of type E_8 and $r_{E_8}(G)$ has odd order, does $\kappa(G)$ belong to $I^8(k)$?*

If the answer is "yes", then the composition $G \mapsto \kappa(G) \mapsto e_8(\kappa(G))$ would give a nontrivial cohomological invariant (with values in $H^8(k, \mathbb{Z}/2\mathbb{Z})$) of groups of type E_8 in the kernel of the Rost invariant.

As far as answering Question 13.9 goes, a priori one only knows that $\kappa(G)$ is in $I^4(k)$. We will say a bit more in §16 below.

14. Calculation of the Killing form

We now compute the Killing form of a group G of type E_8 constructed via (11.2). For a quaternion k-algebra Q_i, we write q_i for the quadratic form $Q_i \to k$ given by the reduced norm and q'_i for the unique 3-dimensional form such that $\langle 1 \rangle \oplus q'_i$ is isomorphic to q_i.

THEOREM 14.1. *Let G be a group of type E_8 constructed from quaternion algebras Q_1, Q_2, Q_3, Q_4 and $c \in k^\times/k^{\times 2}$ as in* (11.2)*. Then the reduced Killing form of G is isomorphic to*

$$8\langle c \rangle \oplus \langle -2 \rangle \langle 1, -c \rangle \left(\sum_i q'_i \oplus \sum_{i<j<\ell} q'_i q'_j q'_\ell \right).$$

We use the reduced Killing form instead of the usual one, in order to get a nontrivial result in characteristics 3 and 5. For most of the proof, we assume that k has characteristic zero; we show in 14.8 below that the result transfers to prime characteristic.

We prove the theorem by restricting the adjoint representation $\mathfrak{g}$ of G to the subgroup $\mathrm{HSpin}\,(A, \sigma)$ for $(A, \sigma) = \otimes(Q_i, ^-)$. We can compute the restriction of G to $\mathrm{HSpin}\,(A, \sigma)$ over an algebraic closure, where we find that $\mathfrak{g}$ is a direct sum of the adjoint representation of HSpin_{16} and the natural half-spin representation [**MP**, p. 305]. As the reduced Killing form $\mathrm{redKill}_G$ is invariant under HSpin_{16}, it follows that these two irreducible representations of HSpin_{16} in $\mathfrak{g}$ are orthogonal relative to $\mathrm{redKill}_G$. We compute the restriction of the reduced Killing form to each summand separately.

14.2. Killing form of HSpin (A, σ). The Lie algebra of HSpin (A, σ) can be identified with the space of σ-skew-symmetric elements in A. Because (A, σ) is a tensor product of quaternion algebras, this space can be described inductively as a tensor product of the subspaces of the Q_i that are symmetric or skew-symmetric under $^-$. (It is clear that such tensor products can be formed that belong to the skew elements in A, and dimension count shows that all skew elements of A are obtained in this way.) The trace quadratic form $x \mapsto \mathrm{tr}_{Q_i}(x^2)$ restricts to be $\langle 2 \rangle$ on the $^-$-symmetric elements and $\langle -2 \rangle q_i'$ on the $^-$-skew-symmetric elements. We conclude that the form $a \mapsto \mathrm{tr}_A(a^2)$ on A restricts to

$$\langle -1 \rangle \left(\sum_i q_i' \oplus \sum_{i<j<\ell} q_i' q_j' q_\ell' \right) \tag{14.3}$$

on the σ-skew-symmetric elements.

The form (14.3) is invariant under HSpin (A, σ), so it is a scalar multiple of the Killing form. By [**Bou**, chap. VIII, §13, Exercise 12], the Killing form of HSpin (A, σ) is $\langle h_{D_8} \rangle$ times the form $a \mapsto \mathrm{tr}_A(a^2)$ from (14.3), where h_{D_8} denotes the Coxeter number, which is 14.

Note that for $X_\alpha, X_{-\alpha}$ belonging to our fixed pinning of E_8 and spanning the highest and lowest root subalgebras of HSpin_{16}, we have $\mathrm{Kill}_{\mathrm{HSpin}_{16}}(X_\alpha, X_{-\alpha}) = 2h_{D_8}$, but $\mathrm{Kill}_{E_8}(X_\alpha, X_{-\alpha}) = 2h_{E_8}$ by [**SS**, pp. E-14, E-15], where h_{E_8} is the Coxeter number of E_8, i.e., 30. We conclude that the restriction of the Killing form of E_8 to the adjoint representation of HSpin (A, σ) is $\langle h_{E_8} \rangle$ times the form (14.3), i.e., the reduced Killing form of G restricts to be $\langle 2 \rangle$ times (14.3) on the Lie algebra of HSpin_{16}.

14.4. Half-spin representation. We restrict the half-spin representation of HSpin (A, σ) to the product of the $\mathrm{PGL}(Q_i)$'s. Putting ω_i for the unique fundamental dominant weight of $\mathrm{PGL}(Q_i)$, the half-spin representation decomposes as a direct sum of the four 5-dimensional representations with highest weight $4\omega_i$ and the four 27-dimensional representations with highest weight $2\omega_i + 2\omega_j + 2\omega_\ell$ with $i < j < \ell$.

Now the representations of $\mathrm{PGL}(Q_i)$ with highest weights $2\omega_i$ and $4\omega_i$ support $\mathrm{PGL}(Q_i)$-invariant quadratic forms isomorphic to

$$\langle 2 \rangle q_i' \quad \text{and} \quad \langle 2 \rangle q_i \oplus \langle 6 \rangle$$

respectively by [**Ga 08**]. Because the Q_i are interchangeable (Prop. 11.5), the half-spin representation contributes

$$\langle 2 \rangle \langle c\, m_2 \rangle \sum_{i<j<\ell} q_i' q_j' q_\ell' \oplus \langle c\, m_4 \rangle \sum_i (\langle 2 \rangle q_i \oplus \langle 6 \rangle) \tag{14.5}$$

to the reduced Killing form of G, where m_2, m_4 are elements of $k^\times$ that are not yet determined. (The factor of c appears here because the center of HSpin_{16} acts as ± 1 on the half-spin representation.)

14.6. Combining the results of 14.2 and equation (14.5), we find that the reduced Killing form of G is

$$8 \langle c\, m_4 \rangle \oplus \langle 2 \rangle \langle -1, c\, m_4 \rangle \sum q_i' \oplus \langle 2 \rangle \langle -1, c\, m_2 \rangle \sum q_i' q_j' q_\ell'. \tag{14.7}$$

(Here we have used the fact that $4\langle 2,6\rangle$ is isomorphic to $8\langle 1\rangle$; as both forms belong to $I^3\mathbb{Q}$, the isomorphism is obvious by the Hasse-Minkowski Principle.)

We now compute m_4 up to sign, for which it suffices to consider the case where the Q_i's are split and $c = 1$. As the Q_i's are interchangeable, we focus on Q_1. The only root of E_8 that restricts to $4\omega_1$ is $-(\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4 + \varepsilon_5)$. We write X_1 for the element of the Chevalley basis of E_8 corresponding to that root and X_{-1} for the element corresponding to the negation of the root; X_1 and X_{-1} are highest and lowest weight vectors respectively for the irreducible representation of $\mathrm{PGL}_2^{\times 4}$ with highest weight $4\omega_1$. We have

$$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) X_1 = \pm X_{-1}$$

because all the roots in E_8 have the same length. Therefore,

$$\mathrm{redKill}_{E_8}(X_1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) X_1) = \pm\mathrm{redKill}_{E_8}(X_1, X_{-1}) = \pm 1.$$

On the other hand, for f the symmetric bilinear form on the representation of $\mathrm{PGL}(Q_1)$ with highest weight $4\omega_1$ such that f is isomorphic to $\langle 2\rangle q_1 + \langle 6\rangle$, we have

$$f(X_1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) X_1) = 1,$$

see [**Ga 08**, 2.4]. Consequently, $m_4 = \pm 1$. Similarly, m_2 is also 1 or -1, possibly with a different sign from m_4.

We claim that m_2 and m_4 equal 1. As they are ± 1 and are defined over $\mathbb{Z}$, it suffices to check that they map to 1 in $\mathbb{R}^\times/\mathbb{R}^{\times 2}$. Consider the case $k = \mathbb{R}$ and $c = 1$ and where exactly one of the Q_i is nonsplit. Computing in the Witt ring, we have:

$$\sum q'_i = 0 \quad \text{and} \quad \sum q'_i q'_j q'_\ell = 8.$$

Plugging this data into (14.7), we find that the reduced Killing form equals $8\langle m_2, m_4\rangle - 8$. On the other hand, the group is isotropic and has Rost invariant zero, so it is split and the reduced Killing form equals 8. It follows that $\langle m_2, m_4\rangle$ equals 2, i.e., $m_2 = m_4 = 1$. This completes the proof of Th. 14.1 in case k has characteristic zero.

14.8. Now suppose that k has characteristic ≥ 3. Fix a field K of characteristic zero that is complete with respect to a discrete valuation with residue field k. The map (11.2) is defined over $\mathbb{Z}$, so it is compatible with the natural maps for lifting k-torsors to K-torsors. We obtain a commutative diagram:

$$\begin{array}{ccccc} H^1(k, \mathrm{PGL}_2^{\times 4} \times \boldsymbol{\mu}_2) & \longrightarrow & H^1(k, E_8) & \xrightarrow{\mathrm{redKill}} & W(k) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(K, \mathrm{PGL}_2^{\times 4} \times \boldsymbol{\mu}_2) & \longrightarrow & H^1(K, E_8) & \xrightarrow{\mathrm{redKill}} & W(K) \end{array}$$

where the vertical arrow between the Witt rings is defined by sending the 1-dimensional form $\langle x\rangle$ for $x \in k^\times$ to the form $\langle u\rangle$, where $u \in K^\times$ is a unit with residue x [**Lam**, §VI.1]. As this map is an injection and the theorem holds over K, the theorem also holds over k. □

Example 14.9. As a corollary of the proof, we can see that in case q is a 4-Pfister quadratic form, there is a $\mathrm{Spin}(q)$-invariant quadratic form on each half-spin representation that is isomorphic to $8q$. Indeed, we can write q as a product $q_1 q_2$ of 2-Pfister forms, and let Q_i be the quaternion algebra with norm q_i. The tensor product $(Q_i, {}^-) \otimes (Q_i, {}^-)$ is $M_4(k)$ endowed with an orthogonal involution adjoint to

q_i. Setting $Q_3 = Q_1$ and $Q_4 = Q_2$ and evaluating (14.5) with $c = \mu_2 = \mu_4 = 1$ gives the claim.

15. The Killing form and E_8's arising from (11.2)

This section gives some properties of the groups of type E_8 constructed via (11.2), proved using the calculation of the Killing form in the preceding section.

EXAMPLE 15.1. The classification of groups of type E_8 over the real numbers was recalled in Example 13.4. Such groups are distinguished (up to isomorphism) by their Killing forms. We now observe that construction (11.2) produces all three groups. There are only two possibilities (split or not) for each of the four quaternion algebras as well as for c.

c	# of nonsplit quaternion algebras	signature of Killing form	description of group
1	0 through 4	8	split
-1	0 or 2	8	split
-1	1 or 3	-24	isotropic
-1	4	-248	anisotropic/compact

Chernousov's Hasse Principle [**PR**, p. 286, Th. 6.6] for E_8 implies that construction 11.2 produces every group of type E_8 over a number field. (In fact, the same reasoning shows that Tits's construction also gives every group of type E_8 over a number field, as was observed already by Ferrar [**F**]. With this in mind, one can view Chernousov's result as saying that E_8's over number fields are much less complicated than groups of type E_8 in general.)

Recall the definition of $\kappa(G)$ from 13.3; it measures the difference of the reduced Killing form of G from the same form for the split E_8.

PROPOSITION 15.2. *For a group G of type E_8 constructed from quaternion algebras Q_1, Q_2, Q_3, Q_4 and $c \in k^\times/k^{\times 2}$ via* (11.2)*, we have*

$$\kappa(G) = \langle\!\langle c \rangle\!\rangle \left[4 \sum_i q_i - 2 \sum_{i<j} q_i q_j + \sum_{i<j<\ell} q_i q_j q_\ell \right] \quad \in I^5.$$

We use the notation $\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ for the tensor product $\langle 1, -x_1 \rangle \otimes \cdots \otimes \langle 1, -x_n \rangle$. As in the previous section, q_i denotes the reduced norm on Q_i.

PROOF OF PROP. 15.2. Example 13.1 and Th. 14.1 give

$$\kappa(G) = \langle\!\langle c \rangle\!\rangle \left(8 + \sum_i q'_i + \sum_{i<j<\ell} q'_i q'_j q'_\ell \right).$$

Replacing each q'_i with $q_i - 1$ and expanding gives the displayed formula. The form is obviously in I^5. □

15.3. It is easy to see that I^5 in Prop. 15.2 "cannot be improved", i.e., that $\kappa(G)$ need not lie in I^6. One can take k to be $\mathbb{R}$ with indeterminates x, y, c adjoined, and put $Q_1 := (x, y)$ and Q_2, Q_3, Q_4 split. The resulting G has $\kappa(G) = 4\langle\!\langle c, x, y \rangle\!\rangle$.

Similarly, if -1 is a square in k, then $2 = 0$ in the Witt ring and $\kappa(G) = \langle\!\langle c \rangle\!\rangle \sum q_i q_j q_\ell$ belongs to I^7. Again, this cannot be improved, as can be seen by taking Q_1, Q_2, Q_3 to be "generic" quaternion algebras, and Q_4 to be split.

However, the Killing forms of the groups G constructed from (11.2) are mainly of interest in case the Rost invariant $r(G)$ is zero. That case is treated by the following theorem provided by Detlev Hoffmann.

THEOREM 15.4 (Hoffmann). *Let G be a group of type E_8 constructed via* (11.2) *from quaternion algebras Q_1, Q_2, Q_3, Q_4 and $c \in k^\times$. If $r(G) = 0$, then*

$$\kappa(G) = 2\langle\langle c\rangle\rangle q_1 q_2 q_4 \quad \in I^8.$$

The theorem says that for groups of type E_8 arising from construction (11.2), the answer to Question 13.9 is "yes".

PROOF OF TH. 15.4. The hypothesis implies that $\langle\langle c\rangle\rangle \sum q_i$ belongs to I^4, hence that $\langle\langle c\rangle\rangle(q_1' - q_2')$ and $\langle\langle c\rangle\rangle(q_3' - q_4')$ are congruent mod I^4, so [**H**, Cor.][c] gives that

$$\langle\langle c\rangle\rangle(q_1 - q_2) = \langle m\rangle\langle\langle c\rangle\rangle(q_3 - q_4) \text{ for some } m \in k^\times.$$

This implies that

$$\langle\langle c\rangle\rangle(q_1 - q_2)^2 = \langle\langle c\rangle\rangle(q_3 - q_4)^2 \tag{15.5}$$

in the Witt ring. Further,

$$\langle\langle c\rangle\rangle q_1 q_2 (q_3 - q_4) = \langle m\rangle\langle\langle c\rangle\rangle q_1 q_2 (q_1 - q_2) = 0 \tag{15.6}$$

and similarly

$$\langle\langle c\rangle\rangle q_1 (q_3 - q_4)^2 = \langle\langle c\rangle\rangle q_1 (q_1 - q_2)^2 = 4\langle\langle c\rangle\rangle q_1 (q_1 - q_2). \tag{15.7}$$

Of course, the roles of q_1 and q_2 are not special, and the same identities hold for every permutation of the subscripts.

We now compute:

$$\begin{aligned}\langle\langle c\rangle\rangle \left(4\sum q_i - 2\sum q_i q_j\right) &= \langle\langle c\rangle\rangle(\sum q_i^2 - 2\sum q_i q_j) \\ &= \langle\langle c\rangle\rangle \left(\sum_{i<j}(q_i - q_j)^2 - 2\sum q_i^2\right).\end{aligned}$$

Applying (15.5), we find:

$$\begin{aligned}\langle\langle c\rangle\rangle \left(4\sum q_i - 2\sum q_i q_j\right) &= \langle\langle c\rangle\rangle \left(2(q_1 - q_2)^2 + 2(q_1 - q_3)^2 + 2(q_1 - q_4)^2 - 2\sum Q_i^2\right) \\ &= \langle\langle c\rangle\rangle \left[4q_1^2 - q_1(4q_2 + 4q_3 + 4q_4)\right].\end{aligned}$$

We can replace the $4q_3 + 4q_4$ with $(q_3 - q_4)^2 + 2q_3 q_4$, and further replace that with $4q_1 - 4q_2 + 2q_3 q_4$ by (15.7), i.e.,

$$\begin{aligned}\langle\langle c\rangle\rangle \left(4\sum q_i - 2\sum q_i q_j\right) &= \langle\langle c\rangle\rangle \left[4q_1^2 - 4q_1 q_2 - 4q_1^2 + 4q_1 q_2 - 2q_1 q_3 q_4\right] \\ &= -2\langle\langle c\rangle\rangle q_1 q_3 q_4.\end{aligned}$$

Evaluating the full Killing form, we have:

$$\begin{aligned}\kappa(G) &= \langle\langle c\rangle\rangle(q_1 q_2 q_3 - q_1 q_3 q_4 + q_2 q_3 q_4 + q_1 q_2 q_4) \\ &= \langle\langle c\rangle\rangle \left[q_1 q_2 (q_3 - q_4) + 2q_1 q_2 q_4 - (q_1 - q_2) q_3 q_4\right],\end{aligned}$$

and the claim follows by applying (15.6) twice. ☐

[c]The statement of this result includes the hypothesis that the 12-dimensional form $\langle\langle c\rangle\rangle(q_1' - q_2')$ is anisotropic, but it is unnecessary.

Of course, the roles of Q_1, Q_2, Q_4 in the theorem are not special, and one can take any three of the four quaternion algebras by (15.6).

REMARK 15.8. One can view Th. 15.4 as giving a relationship amongst four symbols in $H^d(k, \mathbb{Z}/2\mathbb{Z})$ that sum to zero. For three symbols, one has the Elman-Lam Linkage Theorem [**Lam**, X.6.22].

EXAMPLE 15.9. Suppose that k is a field with $2^3 \cdot I^5 \neq 0$ in the Witt ring; for example, this happens if k is formally real. Then there is a 5-Pfister form ϕ such that $2^3\phi \neq 0$. We write $\phi = \langle\langle c \rangle\rangle q_1 q_2$ for 2-Pfister forms q_1, q_2, and $c \in k^\times$. Taking

- construction (11.2) where Q_1, Q_2, Q_3, Q_4 have norms q_1, q_2, q_1, q_2 respectively, or
- Tits's construction 11.6 with a reduced Albert algebra such that $\gamma_3 = \phi_3 = \langle\langle c \rangle\rangle q_1$ and $\phi_5 = \phi$,

one obtains a group G of type E_8 over k that has zero Rost invariant, has $\kappa(G) = 8\phi$ nonzero, and is anisotropic (by Proposition 12.1(1)).

For $k = \mathbb{R}$ or a number field with a unique real embedding, the form ϕ is $\langle\langle -1 \rangle\rangle^5$ and G is such that $G_{\mathbb{R}}$ is compact.

16. A conjecture, and its consequences

Consider the following statement:

(16.1) *For every odd-degree separable extension K/k and every central simple K-algebra of degree 16 with orthogonal involution (A, σ) in I^3K, we have: If $e_3^{16}(A, \sigma)$ is zero, then there is an odd-degree separable extension L/K such that $(A, \sigma) \otimes L$ is completely decomposable.*

By Prop. 2.9, if $e_3^{16}(A, \sigma)$ is zero, then (A, σ) is generically Pfister. That is, (16.1) is somewhat weaker than a "yes" answer to Question 1.3. It is natural to hope that (16.1) holds for every field k of characteristic different from 2.

We have:

THEOREM 16.2. *Suppose that k is a field of characteristic $\neq 2$ for which* (16.1) *holds. Then for every group G of type E_8 over k such that $r_{E_8}(G)$ has odd order, we have:*

(1) *$\kappa(G)$ belongs to I^8k.*
(2) *There is an odd-degree separable extension L/k such that $\mathrm{res}_{L/k}\, G$ is in the image of the map*

$$H^1(L, \mathrm{PGL}_2^{\times 4} \times \boldsymbol{\mu}_2) \to H^1(L, E_8)$$

defined in §11.

If (16.1) holds for k, then part (1) of the theorem says that the answer to Question 13.9 is "yes" and part (2) says that we know all the groups of type E_8 in the kernel of the Rost invariant up to odd-degree extensions of k.

PROOF OF THEOREM 16.2. We first prove (2). Note that there is a separable extension K/k of odd degree such that $\mathrm{res}_{K/k}\, G \in H^1(K, E_8)$ is the image of some $\eta \in H^1(K, \mathrm{HSpin}_{16})$. This follows from the fact that HSpin_{16} contains a maximal torus of E_8 and that the 2-Sylow subgroups of the Weyl groups have order 2^{14} in both cases, see e.g. the proof of [**Ga 09**, 13.7]. (For HSpin_{16}, one checks that the 2-primary part of $2^7\, 8!$ is 2^{14}.)

We enlarge K so that it also kills $r(G)$. By (16.1), there is an odd-degree extension L/K such that the image of η in $H^1(L, \mathrm{PSO}_{16})$ is the class of a tensor product $\otimes_{i=1}^4 (Q_i, {}^-)$, and it follows that $\mathrm{res}_{L/k} G$ is in the image of $H^1(L, \mathrm{PGL}_2^{\times 4} \times \boldsymbol{\mu}_2)$.

We now prove (1); let L be as in (2). As G arises from construction (11.2) over L, $\kappa(G)$ is in $I^8 L$ by Th. 15.4. The class $e_4(\kappa(G)) \in H^4(k, \boldsymbol{\mu}_2)$ is killed by L and so is zero, hence $\kappa(G)$ is in $I^5 k$. Repeating this with e_5, e_6, and e_7 shows that $\kappa(G)$ is in $I^8 k$. □

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Appendix A. Non-hyperbolicity of orthogonal involutions By Kirill Zainoulline

The purpose of the following notes is to prove the following

PROPOSITION A.1. *Let (A, σ) be a central simple algebra with orthogonal involution over a field k of characteristic $\neq 2$. If $\deg A/\operatorname{ind} A$ is odd, then the involution σ is not hyperbolic over the function field of the Severi-Brauer variety of A.*

This result can also be deduced from the divisibility of the Witt index of (A, σ) proved recently by N. Karpenko (see [**Ka I**, Th. 3.3]). Our arguments use the notion of the J-invariant instead.

PROOF OF PROP. A.1. The case where A has index 1 is clear and the index 2 case is [**PSS**, Prop. 3.3], so we may assume that A has index at least 4 and hence the degree is divisible by 4. Further, we may assume that (A, σ) is in I^3 as in Example 1.2(2), otherwise the conclusion is obvious.

Consider the groups $G = \mathrm{HSpin}(A, \sigma)$ and $G' = \mathrm{PGL}(A)$. Let X and X' be the respective varieties of Borel subgroups.

(A.2) Assume that σ is hyperbolic over the function field $k(SB(A))$.

Then the group G is split over $k(SB(A))$ and, hence, over $k(X')$. Since the group G is split over $k(X)$, the algebra A and the group G' are split over $k(X)$.

By the main result of the paper [**PSZ**] (Theorems 4.9 and 5.1) to any simple linear algebraic group of inner type over k and its torsion prime p one may associate an indecomposable Chow motive $\mathcal{R}_p$ such that over the algebraic closure $\bar{k}$ of k the generating function of $\mathcal{R}_p$ is given by the product of r cyclotomic polynomials

$$\prod_{i=1}^{r} \frac{1 - t^{d_i 2^{j_i}}}{1 - t^{d_i}}, \text{ where } 0 \le j_i \le k_i \text{ and } d_i > 0$$

and the explicit values of the parameters d_i and bounds k_i are provided in [**PSZ**, Table 6.3]. The r-tuple of integers $(j_1, j_2, \ldots, j_r)$ is called the J-invariant.

Let $\mathcal{R}_2(G)$ and $\mathcal{R}_2(G')$ be the respective motives for the groups G and G' and for $p = 2$. By [**PSZ**, Prop. 5.3] applied to G over $k(X')$ and G' over $k(X)$ we obtain the following motivic reformulation of the assumption (A.2):

(A.3) $$\mathcal{R}_2(G) \simeq \mathcal{R}_2(G').$$

Since the group G' is a twisted form of the group $\mathrm{PGL}_{\deg A}$, by the first line of [**PSZ**, Table 6.3] the J-invariant of G' has only one entry ($r = 1$), and the parameter d_1 is 1. Then by the proof of [**PSZ**, Lemma 7.3], we obtain that the J-invariant is the list consisting of the single element s, where 2^s is the index of A. Hence the generating function of $\mathcal{R}_2(G')$ is $(1 - t^{2^s})/(1 - t)$.

Similarly, since the group G is a twisted form of the group $\mathrm{HSpin}_{\deg A}$, by [**PSZ**, Table 6.3] the J-invariant of G has $\frac{1}{4}\deg A$ entries with $d_i = 2i-1$ and the following inequality holds

$$j_1 \leq k_1 = v_2(\tfrac{1}{2}\deg A) = v_2(2^{s-1} \cdot \tfrac{\deg A}{\operatorname{ind} A}) = s - 1 < s,$$

where v_2 is the 2-adic valuation. (Here we essentially use that $\frac{\deg A}{\operatorname{ind} A}$ is odd.)

The isomorphism (A.3) implies the equality of the respective generating functions, namely

$$\frac{1-t^{2^s}}{1-t} = \prod_{i=1}^{s/2} \frac{1 - t^{(2i-1)2^{j_i}}}{1 - t^{2i-1}}\ , \text{ where } j_1 < s.$$

We claim that it never holds. Indeed, comparing the coefficients at t^2 and t^3 of the polynomials at the left and the right hand side, we conclude that $j_1 \geq 2$ and $j_2 = 0$. Then comparing them consequently at powers t^{2i-2} and t^{2i-1}, $i \geq 3$ we conclude that $2^{j_1} \geq 2i - 1$ and $j_i = 0$. Therefore, $j_3 = \cdots = j_{s/2} = 0$ and j_1 must coincide with s, which is not the case, since $j_1 < s$.

Hence, the assumption (A.2) fails and the lemma is proven. □

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