DEGREE 5 INVARIANT OF $E_8$

SKIP GARIBALDI AND NIKITA SEMENOV

Abstract. We give a formula for the recently-discovered degree 5 cohomological invariant of some groups of type $E_8$ and use this formula to give a precise interpretation of Serre’s “funny-looking statement” in terms of embeddings finite subgroups in the split $E_8$.

1. Introduction

Let $G$ be a split simple linear algebraic group over a field $k$ of characteristic 0. One of the main goals of the theory of linear algebraic groups over arbitrary fields is to compute the Galois cohomology set $H^1(k, G)$.

One of the main tools was suggested by J-P. Serre in the 1990s, namely the Rost invariant

$$r_G : H^1(\ast, G) \to H^3(\ast, \mathbb{Q}/\mathbb{Z}(2))$$

discovered by M. Rost and explained in Merkurjev’s portion of the book [GMS]. It is a morphism of functors from the category of fields over $k$ to the category of pointed sets.

Mimicking the situation in topology one can consider the kernel of the Rost invariant and try to define a cohomological invariant on it. In the theory of quadratic forms this procedure leads to the invariants defined on the powers of the fundamental ideal $I^n$.

In the present paper we consider the most complicated and yet unsettled case when $G$ has Cartan-Killing type $E_8$. In the first part of the paper we recall the recently-discovered invariant $u$ defined on the kernel of the Rost invariant for groups of type $E_8$ and find an explicit formula of $u(G)$ for groups $G$ obtained by a Tits construction. This formula leads to several surprising results—an answer to a question of Serre about Killing forms (Remark 3.14), cohomological invariants of Spin$_{16}$ (Section 4), the kernel of the Rost invariant for some nonsplit groups of type $E_7$ (Section 5), and finite subgroups of algebraic groups (Section 6).

It turns out that under some additional conditions cohomological invariants provide an obstruction for certain finite groups to be subgroups of algebraic groups. This is connected with Serre’s “funny-looking statement”

2000 Mathematics Subject Classification. Primary 11E72; secondary 17B25, 20C33, 20G15.
from [GR, p. 209]:

“Let $K$ be a field of characteristic 0, and $G$ a group of type $E_8$ over $K$. Suppose that $SL_2(32)$ can be embedded in $G(K)$. Then $PGL_2(31)$ can be embedded in $G(K)$. Nice!”

More precisely, Serre proved that $PGL_2(31)$ can be embedded in $G(K)$ iff $G$ “is compact”, i.e., isomorphic to the scalar extension of the anisotropic $E_8$ over $\mathbb{Q}$, and that $SL_2(32)$ embeds in $G(K)$ iff $G$ is compact and $\cos(2\pi/11)$ is in $K$. This led him to the question: How to tell, e.g., if the split $E_8$ is compact in this sense? We show:

1.1. **Theorem.** $PGL_2(31)$ embeds in $E_8(K)$ if and only if $-1$ is a sum of 16 squares in $K$. The group $SL_2(32)$ embeds in $E_8(K)$ if and only if $-1$ is a sum of 16 squares in $K$ and $\cos(2\pi/11)$ is in $K$.

**2. Preliminaries**

Let $k$ denote a field of characteristic 0. We write $E_8$ for the split simple algebraic group with Killing-Cartan type $E_8$. The Galois cohomology set $H^1(k,E_8)$ classifies simple algebraic groups of type $E_8$ over $k$.

We put

$$H^1(k,E_8)_0 := \{ \eta \in H^1(k,E_8) \mid r_{E_8}(\eta) \text{ has odd order} \}.$$ 

In [Sem 08, Corollary 8.7], the second author defined a morphism of functors:

$$u: H^1(\ast,E_8)_0 \to H^5(\ast,\mathbb{Z}/2\mathbb{Z}).$$

This is the degree 5 invariant from the title.

Let now $G$ be a group of type $E_8$. It corresponds with a canonical element of $H^1(k,E_8)$, so it makes sense to speak of “the Rost invariant of $G$”; we denote it by $r(G)$. Suppose now that $r(G)$ has odd order, so $G$ belongs to $H^1(k,E_8)_0$. The second author also proved in [Sem 08]:

(2.1) $u(G) = 0$ if and only if there is an odd-degree extension of $k$ that splits $G$.

For example, the compact group $G$ of type $E_8$ over $\mathbb{R}$ has Rost invariant zero and $u(G) = (-1)^5$.

As an obvious corollary, we have:

(2.2) If $k$ has cohomological dimension $\leq 2$, then every $k$-group of type $E_8$ is split by an odd-degree extension of $k$.

(This is the $p = 2$ case of [Gi01, p. 302, Th. 4(2)].) Serre’s “Conjecture II” for groups of type $E_8$ is that in fact every group of type $E_8$ over such a field is split.
3. TIT’S CONSTRUCTION OF GROUPS OF TYPE $E_8$

3.1. In [T]—or see [J]—Jacques Tits gave an explicit construction that takes an octonion $k$-algebra and an Albert $k$-algebra as inputs, where $k$ is any field of characteristic $\neq 2, 3$, and gives as output a Lie algebra of type $E_8$. (Really, Tits’s construction is more general and gives other kinds of groups as well. The variety of possibilities is summarized in Freudenthal’s magic square as in [Inv, p. 540]. However, the flavor in all cases is the same, and this case is the most interesting; see also Section 5.) As his construction is obviously functorial in $k$ and compatible with automorphisms of the inputs, one finds an injective homomorphism $G_2 \times F_4 \to E_8$ where we have written the Killing-Cartan type for the corresponding split group. Applying Galois cohomology, we have a morphism of functors

\[ H^1(\ast, G_2) \times H^1(\ast, F_4) \to H^1(\ast, E_8). \]

For a given field $k$, the three sets classify octonion $k$-algebras, Albert $k$-algebras, and simple Lie algebras of type $E_8$ respectively; by Galois descent it is obvious that this map coincides with the construction described by Tits.

3.3. Here is an alternative derivation of (3.2) that avoids Tits’s paper [T] but assumes that $k$ has characteristic zero (as we will anyway assume in the rest of the paper). The root system of $E_8$ contains a maximal subsystem of type $E_6 \times A_2$ corresponding to a copy of $(E_6^{sc} \times SL_3)/\mu_3$ in the group $E_8$, where $E_6^{sc}$ denotes the split simply connected group of type $E_6$. The subgroup of $E_6^{sc}$ fixed by the obvious automorphism of the Dynkin diagram (viewed as an automorphism of a pinning of $E_6^{sc}$) is split of type $F_4$, see e.g. [Ga 09a, 9.12]. The centralizer of this $F_4$ is a simple group of type $G_2$ (because this is so over $\mathbb{C}$; the $F_4$ and $G_2$ subgroups form a dual pair) and is split because it contains a copy of $SL_3$. This gives a subgroup $G_2 \times F_4$ of $E_8$ over $k$ and applying Galois cohomology gives a morphism as in (3.2).

But note that the two derivations of (3.2) give the same morphism. Indeed, write $\phi$, $\phi'$ for the embeddings of $G_2 \times F_4$ in the split $E_8$ over $k$. For $\overline{k}$ an algebraic closure of $k$, there exists some $g \in E_8(\overline{k})$ such that $g(\text{im } \phi)g^{-1} = \text{im } \phi'$ because this is so over $\mathbb{C}$ [D]. Further, the composition $(\phi')^{-1} \text{Int}(g)\phi$ is a $\overline{k}$-automorphism of $G_2 \times F_4$ (where $\text{Int}(g)y := gyg^{-1}$), hence is $\text{Int}(x)$ for a unique $x \in (G_2 \times F_4)(\overline{k})$. That is, $\text{Int}(g)\phi = \phi'\text{Int}(x)$. Replacing $g$ with $\phi'(x)^{-1}g$, we may assume that $\text{Int}(g)\phi = \phi'$. For $\sigma$ in the absolute Galois group $\text{Gal}(\overline{k}/k)$ of $k$, we have:

\[ \text{Int}(g)\phi = \phi' = \sigma \phi' = \text{Int}(\sigma g)\phi \]

because $\phi$ and $\phi'$ are $k$-defined, hence $g^{-1}\sigma g$ centralizes $\text{im } \phi$, i.e., $g^{-1}\sigma g = 1$ and $g \in E_8(k)$. It is now obvious that the morphisms (3.2) induced by $\phi$ and $\phi'$ are equal.

3.5. Remark. It follows quickly from the discussion in 3.3 that a group $G_0$ of type $E_8$ is obtained by Tits’s construction—meaning, is in the image of
Following [Inv], we write 3.6. from Tits’s construction. We do this in Theorem 3.10. Rost invariant of odd order (so that it makes sense to speak of type $F$ invariant of an Albert algebra or an octonion algebra (equivalently, a group invariant of an Albert algebra; such algebras also have an invariant subgroup of $E$ $(3.2)$—)

...4 SKIP GARIBALDI AND NIKITA SEMENOV...5 points as $E_8$, but a different Galois action $*$, namely $\sigma \ast g = \eta_\sigma(\sigma \cdot g)\eta_\sigma^{-1}$ for $g \in G_0(\mathcal{F})$. As $G_0$ contains a subgroup of type $G_2 \times F_4$ and all such subgroups of $E_8$ are conjugate over $\bar{k}$, we may replace $\eta$ by a cohomologous cocycle and so assume that $\eta_\sigma$ normalizes $G_2 \times F_4$ for every $\sigma \in \text{Gal}(\bar{k}/k)$. But $G_2 \times F_4$ has only inner automorphisms and is its own centralizer in $E_8$, so $\eta_\sigma$ belongs to $G_2 \times F_4$ for every $\sigma$, whence $G_0$ is in the image of $(3.2)$.

Our purpose is to compute the value of $u$ on those groups of type $E_8$ with Rost invariant of odd order (so that it makes sense to speak of $u$) and arising from Tits’s construction. We do this in Theorem 3.10.

3.6. Following [Inv], we write $f_3(-)$ for the even component of the Rost invariant of an Albert algebra or an octonion algebra (equivalently, a group of type $F_4$ or $G_2$). We write $g_3(-)$ for the odd component of the Rost invariant of an Albert algebra; such algebras also have an invariant $f_3$ taking values in $H^3(k,\mathbb{Z}/2\mathbb{Z})$. An Albert algebra $A$ has $g_3(A) = 0$ and $f_3(A) = 0$ iff $A$ has a nonzero nilpotent, iff the group $\text{Aut}(A)$ is isotropic.

Suppose now that $G \in H^1(k,E_8)$ is the image of an octonion algebra $O$ and an Albert algebra $A$. It follows from a twisting argument as in the proof of Lemma 5.8 in [GQ]—and was pointed out by Rost as early as 1999—that

$$r(G) = r_{G_2}(O) + r_{F_4}(A).$$

In particular, $G$ belongs to $H^1(k,E_8)_0$ if and only if $f_3(O) + f_3(A) = 0$ in $H^3(k,\mathbb{Z}/2\mathbb{Z})$, i.e., if and only if $f_3(O) = f_3(A)$.

3.7. Definition. Define

$$t: H^1(*,F_4) \to H^1(*,E_8)_0$$

by sending an Albert $k$-algebra $A$ to the group of type $E_8$ constructed from $A$ and the octonion algebra with norm form $f_3(A)$, via Tits’s construction from 3.1. By the preceding paragraph, $r(G) = g_3(A) \in H^3(k,\mathbb{Z}/3\mathbb{Z})$, so $G$ does indeed belong to $H^1(k,E_8)_0$.

3.8. Example. If $A$ has a (nonzero) nilpotent element, then the group $t(A)$ is split. Indeed, $g_3(A)$ is zero so $t(A)$ is in the kernel of the Rost invariant. Also, $t(A)$ is isotropic because it contains the isotropic subgroup $\text{Aut}(A)$, hence $t(A)$ is split by, e.g., [Ga09b, Prop. 12.1(1)].

3.9. Example. In case $k = \mathbb{Q}$ or $\mathbb{R}$, there are exactly three Albert algebras up to isomorphism. All have $g_3 = 0$; they are distinguished by the values of $f_3$ and $f_5$. 
It follows from Chernousov’s Hasse Principle for groups of type $E_8$ [PR] that for every number field $K$ with a unique real place, the set $H^1(K, E_8)_0$ has two elements: the split group and the anisotropic group constructed as in the last line of the table.

3.10. Theorem. For every Albert $k$-algebra $A$, we have:

$$u(t(A)) = f_5(A) \in H^5(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. The composition $ut$ is an invariant $H^1(\ast, F_4) \to H^5(\ast, \mathbb{Z}/2\mathbb{Z})$, hence is given by

$$ut(A) = \lambda_5 + \lambda_2 \cdot f_3(A) + \lambda_0 \cdot f_5(A)$$

for uniquely determined elements $\lambda_i \in H^1(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$, see [GMS, p. 50].

We apply this formula to each of the three lines in the table from Example 3.9. Obviously $u$ of the split $E_8$ is zero, so the first line gives:

$$0 = u(\text{split } E_8) = \lambda_5 \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

Applying this to the second line gives:

$$0 = u(\text{split } E_8) = \lambda_2 \cdot (-1)^3 \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

For the last line, $u$ of the compact $E_8$ is $(-1)^5$ by (2.1), see the end of [Sem08] for details. We find:

$$(-1)^5 = u(\text{compact } E_8) = \lambda_0 \cdot (-1)^5,$$

so $\lambda_0$ equals 1 in $H^0(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

To show that $\lambda_2 = 0$ we proceed as follows. Consider the purely transcendental extension $F = \mathbb{Q}(x, y, z, a, b)$ and let $H$ be the group of type $F_4$ with $f_3(H) = (x, y, z)$, $f_5(H) = f_3(H) \cdot (a, b)$ and $g_3(H) = 0$. Then $ut(H) = f_5(H) + f_3(H) \cdot \lambda_2$.

Let $K$ be a generic splitting field for the symbol $f_5(H)$. Since $H_K$ is isotropic, the resulting group $t(H)$ of type $E_8$ is isotropic over $K$, and, since it has trivial Rost invariant, it splits over $K$ [Ga09b, Prop. 12.1]. Obviously, $ut(H)$ is killed by $K$. Therefore $f_3(H) \cdot \lambda_2$ is zero over $K$. If $f_3(H) \cdot \lambda_2$ is zero over $F$, then by taking residues we see that $\lambda_2$ is zero in $H^2(\mathbb{Q}(a, b), \mathbb{Z}/2\mathbb{Z})$, hence also in $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$. Otherwise, $f_3(H) \cdot \lambda_2$ is equal to $f_5(H)$ by [OViVo, Theorem 2.1], and again completing and taking residues with respect to the $x$-, $y$-, and $z$-adic valuations, we find that $\lambda_2 = (a, b) \in H^2(\mathbb{Q}(a, b), \mathbb{Z}/2\mathbb{Z})$. But this is impossible because $\lambda_2$ is defined over $\mathbb{Q}$. This proves that $\lambda_2 = 0$. \qed

3.11. Corollary. The invariant $u$ is surjective on pure symbols. More precisely, for any field $k$ of characteristic 0 and any pure symbol $v \in H^5(k, \mathbb{Z}/2\mathbb{Z})$ there exists a group of type $E_8$ with zero Rost invariant and with $u(G) = v$. 

<table>
<thead>
<tr>
<th>$f_3(A)$</th>
<th>$f_5(A)$</th>
<th>$t(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>split by Example 3.8</td>
</tr>
<tr>
<td>$(-1)^3$</td>
<td>0</td>
<td>split by Example 3.8</td>
</tr>
<tr>
<td>$(-1)^3$</td>
<td>$(-1)^5$</td>
<td>anisotropic by [J, p. 118]</td>
</tr>
</tbody>
</table>
3.12. Remark. In [Sem 08, pf. of Cor. 8.7] the second author constructed a \( \nu_4 \)-variety \( \tilde{Y} \) that splits \( u \). Voevodsky has conjectured that if \( u \in \text{H}^0(k, \mathbb{Z}/2\mathbb{Z}) \) and \( X \) is a \( \nu_{n-1} \)-variety that splits \( u \), then \( u \) is a pure symbol. Theorem 3.10 confirms this conjecture for the \( E_8' \)'s constructed as in Definition 3.7, equivalently, that have odd-order Rost invariant and contain a subgroup of type \( G_2 \times F_4 \).

For \( G \) in the image of \( t \), \( u(G) \) is a symbol in \( \text{H}^5(k, \mathbb{Z}/2\mathbb{Z}) \), so it determines a 5-Pfister quadratic forms \( q_G \) such that \( e_5(q_G) = u(G) \). We have:

3.13. Corollary. For every field \( k \) of characteristic zero and every group \( G \in \text{H}^1(k, E_8)_0 \) in the image of \( t \), we have:

\[
(60) \langle \text{Kill}_G - \text{Kill}_{E_8} \rangle = 2^3 \cdot q_G \quad \in \text{I}^8(k),
\]

where \( \text{Kill}_- \) denotes the Killing form of \( - \) and \( E_8 \) the split group.

Proof. Combine the formula for \( \text{Kill}_G \) given in [J, p. 117, (114)] or [Ga 09b, 13.2] with Theorem 3.10. \( \square \)

3.14. Remark. The invariant \( u \) was constructed using motives. One might hope to construct it using methods as in [GMS], in particular to deduce \( u(G) \) from the Killing form of \( G \) (suggested by analogy with the way that \( f_5(A) \) is deduced from the trace form on an Albert algebra \( A \) on pages 49–51 of ibid.). But this is impossible. Indeed, fix a formally real field \( k_0 \) and write \( k \) for the function field of the 8-Pfister quadratic form \( \langle -1 \rangle^7 \cdot \langle t_1 t_2 \rangle \) over \( k_0(t_1, t_2) \) where \( t_1 \) and \( t_2 \) are indeterminates. Put \( A_i \) for the reduced Albert algebra over \( k \) with

\[
f_3(A_i) = (-1)^3 \quad \text{and} \quad f_5(A_i) = (-1)^4 \cdot (t_i), \quad i = 1, 2.
\]

Put \( G_i := t(A_i) \); it is a group of type \( E_8 \) over \( k_0(t_1, t_2) \) and \( q_{G_1} = \langle -1 \rangle^4 \cdot \langle t_1 \rangle \) by Theorem 3.10. As \( k_0 \) is formally real, \( q_{G_1} \) and \( q_{G_2} \) are distinct and anisotropic over \( k_0(t_1, t_2) \). For dimension reasons [Lam, X.4.10], \( \langle -1 \rangle^4 \cdot \langle t_1 t_2 \rangle \) is anisotropic over \( k \), hence \( q_{G_1} \) and \( q_{G_2} \) are not isomorphic over \( k \) and in particular \( u(G_1) \neq u(G_2) \) in \( \text{H}^5(k, \mathbb{Z}/2\mathbb{Z}) \). On the other hand, the isomorphism class of \( \langle -1 \rangle^3 q_{G_i} \simeq \langle -1 \rangle^7 \langle t_i \rangle \) over \( k \) does not depend on \( i \), so \( G_1 \) and \( G_2 \) have isomorphic Killing forms over \( k \) by Corollary 3.13.

4. Application to \( \text{Spin}_{16} \)

Recall from [Inv, pp. 436, 437] that the Rost invariant of a class \( \eta \in \text{H}^1(k, \text{Spin}_{16}) \) is given by the formula

\[
r_{\text{Spin}_{16}}(\eta) = e_3(\eta) \quad \in \text{H}^3(k, \mathbb{Z}/2\mathbb{Z})
\]

where \( q_\eta \) is the 16-dimensional quadratic form in \( \text{I}^3 k \) corresponding to the image of \( \eta \) in \( \text{H}^1(k, \text{SO}_{16}) \) and \( e_3 \) is the Arason invariant. It follows that \( \eta \) belongs to the kernel of the Rost invariant if and only if \( q_0 \) belongs to \( \text{I}^k \).

Write \( \text{H}^1(\star, \text{Spin}_{16})_0 \) for the kernel of the Rost invariant; we will determine the cohomological invariants \( \text{H}^1(\star, \text{Spin}_{16})_0 \rightarrow \text{H}^\star(\star, \mathbb{Z}/2\mathbb{Z}) \). To avoid
a long technical digression, we only outline the details of the argument. In [Ga09b, §11], the first author described a subgroup of HSpin$_{16}$ isomorphic to PGL$_2^{\times 4}$. The inverse image of this subgroup in Spin$_{16}$ contains a direct product $V \times V$ of two copies of the Klein four-group so that the image of a quadruple

$$(a_1, a_2, a_3, a_4) \in H^1(K, \mu^4) = H^1(K, V \times V)$$

in $H^1(K, \text{PGL}_2)^{\times 4}$ is the quadruple of quaternion algebras $(a_1, a_2), (a_1, a_2), (a_3, a_4), (a_3, a_4)$. Together with the center of Spin$_{16}$—which is also isomorphic to $V$—we find a copy of $V^{\times 3}$ in Spin$_{16}$ such that the induced map $H^1(K, V^{\times 3}) \to H^1(K, \text{Spin}_{16})$ has image $H^1(K, \text{Spin}_{16})_0$ for every $K/k$.

With this in hand, arguments similar to those in [Ga09a, 18.1, 18.9] give that the invariant $u$ of $H^1(\ast, E_8)_0$ can be used to construct invariants of $H^1(\ast, \text{Spin}_{16})_0$ of degree 5 and 6, and that the collection of invariants $H^1(\ast, \text{Spin}_{16})_0 \to H^\ast(\ast, \mathbb{Z}/2\mathbb{Z})$ is a rank 5 free $H^\ast(k, \mathbb{Z}/2\mathbb{Z})$-module with basis consisting of invariants of degree 0, 4, 5, 5, 6.

From the preceding paragraph, we deduce that the essential 2-dimension of $H^1(\ast, \text{Spin}_{16})_0$ is at least 6. Standard arguments show that the essential dimension is at most 6, so we deduce that the essential dimension and the essential 2-dimension of $H^1(\ast, \text{Spin}_{16})_0$ both equal 6. This should be contrasted with the essential dimension of the functor $H^1(\ast, \text{Spin}_{16})$, which is 24, see [M09, Th. 4.9] and [BRV, Th. 3.3(b)].

5. APPLICATION TO $E_7$

5.1. Example. Put $G$ and $\bar{G}$ for the simply connected and adjoint real forms of $E_7$ with Tits index

$$
\scalebox{1}{
\begin{tikzpicture}
\node (A) at (0,0) {$\circ$};
\node (B) at (1,0) {$\circ$};
\node (C) at (2,0) {$\circ$};
\node (D) at (3,0) {$\circ$};
\node (E) at (4,0) {$\circ$};
\end{tikzpicture}
}$$

i.e., non-split with a minimal parabolic subgroup that is “wesentlich” in the language of [H, p. 132]. Élie Cartan proved that the set $H^1(\mathbb{R}, G)$ has four elements, corresponding to the four real forms of $G$. The image of the natural map $q: H^1(\mathbb{R}, G) \to H^1(\mathbb{R}, \bar{G})$ is the set $\{G, G^{\text{cpt}}\}$ where $G^{\text{cpt}}$ denotes the compact form of $G$; these are the groups whose Tits class is $[H] \in H^2(\mathbb{R}, \mu_2)$. As is standard, $H^1$ of the center of $G$—i.e., $H^1(\mathbb{R}, \mu_2) = \mathbb{Z}/2\mathbb{Z}$—acts transitively on the fibers of $q$. Further, the Rost invariant

$$r_G: H^1(\mathbb{R}, G) \to H^3(\mathbb{R}, \mathbb{Z}/12\mathbb{Z}) = \{0, (-1)^3\} = \mathbb{Z}/2\mathbb{Z}$$

satisfies

$$r_G(z \cdot \eta) = r_G(\eta) + z \cdot (-1)^2$$

for $z \in H^1(\mathbb{R}, \mu_2)$ and $\eta \in H^1(\mathbb{R}, G)$

by [GQ, 2.3] and [Ga01, 7.1]. That is, $q$ is 2-to-1 and each fiber has one element in the kernel of $r_G$ and one element whose Rost invariant is $(-1)^3 \in H^3(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$; in total $H^1(\mathbb{R}, G)$ has four elements. (The description of $H^1(\mathbb{R}, G)$ and the fibers of $q$ can also be obtained from [Serre GC,
III.4.5, Exercise (a)], but that reference does not directly say anything about the Rost invariant.

In particular, the kernel of the Rost invariant \( r_G \) is not zero. This is in contrast with the case of the split simply connected group \( E_7^{sc} \) of type \( E_7 \), where the kernel of the Rost invariant on \( H^1(k,E_7^{sc}) \) is zero for every field \( k \) by [Ga 01].

We will now show how to use the invariant \( u \) to distinguish the classes in the kernel of \( r_G \). Fix a field \( k \) of characteristic zero and a quaternion \( k \)-algebra \( Q \). Define \( G \) to be the simply connected group of type \( E_7 \) whose Tits index is (5.2) (if \( Q \) is division) or is split (if \( Q \) is split). We have:

5.3. Proposition. There are inclusions \( \text{SL}(Q) \times F_4 \subset G \subset E_8 \) such that for \( (\lambda,A) \in H^1(k,\text{SL}(Q)) \times H^1(k,F_4) \), we have:

1. The image of \( (\lambda,A) \) in \( H^1(k,\text{Aut}(G)) \) is a group of type \( E_7 \) whose Lie algebra arises from the Tits construction with \( Q \) and \( A \) as inputs.
2. \( r_G(\lambda,A) = r_{E_8}(\lambda,A) = (\lambda) \cdot [Q] + r_{F_4}(A) \).
3. If \( (\lambda,A) \) is in the kernel of \( r_{E_8} \), then \( u(\lambda,A) = f_5(A) \).

Regarding the statement in (1): Tits’s construction of Lie algebra from \( [T] \), when fed a quaternion algebra \( Q \) and an Albert algebra \( A \) as inputs, produces a Lie algebra (equivalently, a group) of type \( E_7 \) as output. As in 3.1 and 3.3, Tits’s construction amounts to applying \( H^1 \) to an inclusion of groups \( \text{PGL}_2 \times F_4 \subset E_7^{\text{adj}} \), where \( E_7^{\text{adj}} \) denotes the split adjoint group of type \( E_7 \).

5.4. Example. Before we prove the proposition, we apply it to the case \( k = \mathbb{R} \) and \( Q = \mathbb{H} \) from Example 5.1. It follows from the description of real forms of \( E_7 \) in terms of the Tits construction from [J, pp. 120, 121] that the fiber of \( q \) over \( G \) contains the classes \( (\pm 1, \text{split Albert}) \) and the fiber over \( G^{\text{pt}} \) is \( (\pm 1, \text{compact Albert}) \). By 5.3(2), these four classes are distinct and so represent all the elements of \( H^1(\mathbb{R},G) \). The kernel of \( r_G \) consists of \( (1, \text{split}) \) and \( (\pm, \text{compact}) \), and 5.3(3) says that \( u \) sends these two classes to \( 0 \) and \( (\pm 1)^5 \) respectively. That is, \( u \) distinguishes the two classes in \( \ker r_G \), as promised.

For use in the proof of Proposition 5.3, we note that examining the extended Dynkin diagram of \( G_2 \) shows that it contains a maximal subgroup \( D \) isomorphic to \( (\text{SL}_2 \times \text{SL}_2)/\mu_2 \), where the first copy of \( \text{SL}_2 \) has Dynkin index 1 in \( G_2 \) and the second has Dynkin index 3.

5.5. Lemma. For each quaternion \( k \)-algebra \( Q \), there is a unique class \( q \in H^1(k,D) \) whose image in \( H^1(k,G_2) \) is zero and such that \( D \) twisted by \( q \) is isomorphic to \( (\text{SL}_2 \times \text{SL}_2)/\mu_2 \).

Proof. Write \( \phi : \text{SL}_2 \to D \) for the diagonal embedding \( \phi(g) = (g,g) \). Fix a square root \( i \) of \(-1\) in some algebraic closure of \( k \) and put

\[
x := \phi \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) \quad \text{and} \quad y := \phi \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \in D(k(i)).
\]
One checks that $x$ and $y$ are fixed by $\text{Gal}(k(i)/k)$ and so are $k$-points. Further, $x^2 = y^2 = 1$ and $xy = yx$. Therefore, they generate a subgroup of $D$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Write $Q = (a, b)$ for $a, b \in k^\times/k^\times 2 = H^1(k, \mathbb{Z}/2\mathbb{Z})$, and put $q$ for the image of $(a, b) \in H^1(k, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ in $H^1(k, D)$. One checks that each of the two projections $D \rightarrow \text{PGL}_2$ send $q$ to the class of $Q$ in $H^1(k, \text{PGL}_2)$. It follows that $D$ twisted by $q$ is $(\text{SL}(Q) \times \text{SL}(Q))/\mu_2$.

It is an exercise to see that if $Q$ is split, then $q$ is trivial in $H^1(k, D)$. In particular, the image of $q$ under the Rost invariant $H^1(k, G_2) \rightarrow H^3(k, \mathbb{Z}/2\mathbb{Z})$ is a symbol killed by the function field of the Severi-Brauer variety of $Q$, hence is of the form $[Q] \cdot (c)$ for some $c \in k^\times$. Map $\text{SL}(Q)$ into $(\text{SL}(Q) \times \text{SL}(Q))/\mu_2$ by, say, $g \mapsto (g, 1)$ and consider the diagram

$$
\begin{array}{cccc}
H^1(k, D) & \longrightarrow & H^1(k, G_2) & \longrightarrow & H^3(k, \mathbb{Z}/2\mathbb{Z}) \\
\tau_q | & \simeq & \tau_q | & \simeq & \tau_q | \\
H^1(k, \text{SL}(Q)) & \longrightarrow & H^1(k, D_q) & \longrightarrow & H^1(k, (G_2)_q) & \longrightarrow & H^3(k, \mathbb{Z}/2\mathbb{Z})
\end{array}
$$

where $\tau_q$ denotes the twisting isomorphism. The diagram commutes by [Gi00, p. 76, Lem. 7]. The image of $c \in k^\times/\text{Nrd}(Q^\times) = H^1(k, \text{SL}(Q))$ along the bottom row is $[Q] \cdot (c)$ because $\text{SL}(Q)$ has Dynkin index 1 in $D_q$. Replacing $q$ with $\tau_q^{-1}(c)$ gives the desired class. Tracing through the proof also demonstrates the uniqueness of $q$. \hfill \square

**Proof of Prop. 5.3.** Mapping $\text{SL}_2 \rightarrow D$ via $g \mapsto (1, g)$ gives a diagram of inclusions

$$
\begin{array}{cccc}
\text{SL}_2 \times F_4 & \longrightarrow & D \times F_4 & \longrightarrow & G_2 \times F_4 \\
\downarrow & & \downarrow & & \downarrow \\
E_7^{sc} & \longrightarrow & (\text{SL}_2 \times E_7^{sc})/\mu_2 & \longrightarrow & E_8
\end{array}
$$

where the $\text{SL}_2$ in the upper left has Dynkin index 3 in $E_7^{sc}$ and contains the center of $E_7^{sc}$. Twisting this diagram by the class $q$ from Lemma 5.5 gives a diagram of inclusions

$$
\begin{array}{cccc}
\text{SL}(Q) \times F_4 & \longrightarrow & D_q \times F_4 & \longrightarrow & G_2 \times F_4 \\
\downarrow & & \downarrow & & \downarrow \\
G & \longrightarrow & (\text{SL}(Q) \times G)/\mu_2 & \longrightarrow & E_8
\end{array}
$$

(Twisting $E_8$ by $q$ gives back the split $E_8$ because $(E_8)_q$ contains $G_2 \times F_4$, so has $k$-rank at least 6.) Claim (2) follows immediately and Theorem 3.10 gives (3). The image of $\text{SL}_2 \times F_4$ under $E_7^{sc} \rightarrow E_7^{\text{adj}}$ is the subgroup $\text{PGL}_2 \times F_4$ arising in Tits’s construction, giving (1). \hfill \square

Example 5.1 showed that the Rost invariant could have nonzero kernel on $H^1(k, G)$ for $G$ a group with Tits index $(5.2)$. Proposition 5.3 gives more, namely a lower bound on the complexity of this kernel.
5.6. Corollary. For every simply connected $G$ with Tits index (5.2), the functor $H^1(+,G)_0$ has essential (2-)dimension $\geq 3$.

Proof. Put $Q$ for the quaternion division algebra underlying the Tits class of $G$. Suppose first that $cd_2 k \leq 2$. Put $E := k(t_1,t_2,t_3)$ where the $t_i$'s are indeterminates, and write $A$ for the reduced Albert $E$-algebra with $f_3(A) = [Q] \cdot (t_1)$ and $f_5(A) = f_3(A) \cdot (t_2) \cdot (t_3)$. By Prop. 5.3(2), the image $\eta$ of $(t_1,A)$ in $H^1(E,G)$ is in $H^1(E,G)_0$. Further, if $\eta = \res_{E/k}(\eta')$ for some extension $E'$ contained in $E$ and $\eta' \in H^1(E',G)_0$, then $\res_{E'/E}u(\eta') = u(\eta)$, which is the nonzero class $f_5(A)$ in $H^5(E,Z/2Z)$ by Prop. 5.3(3). Hence $cd_2 E' \geq 5$ and $E'/k$ has transcendence degree at least 3. The claim is proved in this case.

For general $k$, the proof of [M92, Th. 4] shows that there is an extension $F/k$ such that $G \times F$ has Tits index (5.2) (because $Q \otimes F$ is division) and $cd_2 F \leq 2$. As the essential 2-dimension can only decrease when we enlarge the base field, the corollary is proved. \hfill \Box

6. GALOIS DESCENT FOR REPRESENTATIONS OF FINITE GROUPS

In this section, we restate some observations of Serre from [Serre00] and [GR] regarding projective embeddings of simple groups in exceptional algebraic groups. Combining these results with the $u$-invariant for $E_8$ gives some new embeddings results, see Example 6.6 below.

Let $A$ be an abstract finite group and $G$ a semisimple linear algebraic group defined over $\mathbb{Q}$. Fix a faithful representation $\pi: G \to \text{GL}_N$ defined over $\mathbb{Q}$.

6.1. Definition. Let $\mathbb{Q} \subset F$ be a field. The character of a homomorphism $\alpha: A \to G(F)$ is the character of the composition $\pi \circ \alpha: A \to \text{GL}_N(F)$. We say that the character of $\alpha$ is defined over $F$ if all its values belong to $F$.

Let $\varphi: A \to G(F)$ be a monomorphism and $\chi$ its character. Assume that $\chi$ is defined over $F$, $Z_{G(F)}(\varphi(A)) = 1$ (in particular, $G$ is adjoint), that there is exactly one $G(F)$-conjugacy class of homomorphisms $A \to G(F)$ with character $\chi$, and $G$ is split or $\text{Aut} G = G$.

The following theorem can be extracted from Serre’s paper [Serre00, 2.5.3].

6.2. Theorem. In the above notation, we have:

1. There exists a unique inner form $G_0$ of $G$ defined over $F$ together with a monomorphism $A \to G_0(F)$ with character $\chi$.
2. If $\psi,\psi': A \to G_0(F)$ have character $\chi$, then there is a unique $g \in G_0(F)$ such that $\psi' = \text{Int}(g) \psi$.
3. For every field extension $K/F$ there is a representation $A \to G(K)$ with character $\chi$ iff $G \simeq G_0$ over $K$.

We recalled the definition of inner form in Remark 3.5. The composition $G(F) \xrightarrow{\pi} \text{GL}_N(F) \xrightarrow{\text{tr}} F$ is invariant under the twisted Galois action $*$ and
so defines a morphism of varieties $G_0 \to \mathbb{A}^1$ over $F$. Using this, it makes sense to speak of the character of a homomorphism $A \to G_0(F)$ as in the theorem.

**Proof.** Let 

$$P = \{\alpha: A \to G \mid \alpha \text{ is a representation with character } \chi\};$$

it is a variety over $F$ and $G$ acts on it by conjugation. By assumptions on $A$ and $G$ this action is transitive. Moreover, the condition on the centralizer guarantees that this action is simply transitive, i.e., for any $\alpha, \beta \in P(\overline{F})$ there exists a unique $g \in G(\overline{F})$ with $\beta = \alpha^g$. Thus, $P$ is a $G$-torsor.

Let $\eta \in H^1(F, G)$ be the 1-cocycle corresponding to the torsor $P$. Then $\sigma \cdot \varphi = \eta^{-1}_\sigma \varphi \eta_\sigma$ for all $\sigma \in \text{Gal}((F)/F)$. Define now $G_0$ as the twisted form of $G$ over $F$ by the torsor $P$, equivalently, the 1-cocycle $\eta$. As $\varphi: A \to G(\overline{F})$ is invariant under the twisted Galois action, it restricts to a homomorphism $A \to G_0(F)$, proving (1).

For (2), we assumed there is a $g \in G_0(F)$ such that $\psi' = \text{Int}(g) \psi$, and this $g$ is in $G_0(F)$ by the argument from (3.4).

Let $K/F$ be a field extension. If there is a representation $A \to G(K)$ with character $\chi$, then obviously $G$ and $G_0$ are isomorphic over $K$. Conversely, if $G$ and $G_0$ are isomorphic over $K$, then the image of the cocycle $\eta$ in $H^1(K, \text{Aut}(G))$ is zero. As $G$ is split adjoint or $\text{Aut} G = G$, it follows that $\eta$ is already zero in $H^1(K, G)$, proving (3). \(\square\)

### 6.3. Example

Trivially, one may replace the conclusion of Theorem 6.2.2 with “there exists $f \in \text{Aut}(G_0)(F)$ such that $\psi' = f \psi$”; one simply takes $f = \text{Int}(g)$. However, this $f$ need not be unique, as we now illustrate. Take $A, G, \varphi, \pi$ as in Th. 6.2 with $G$ split and suppose that the representation $\pi \varphi$ of $A$ is irreducible (e.g., take $A = \text{PSL}(2, 8)$, $G = G_2$ and $\pi$ the 7-dimensional fundamental representation as in Example 6.5 below). We may apply the theorem also where $G, \varphi, \pi$ are replaced by $G \times G, \varphi \times \varphi, \pi \times \pi$: the claim $Z_{G \times G}(\varphi(A), \varphi(A)) = 1$ is obvious and the irreducibility hypothesis shows that every homomorphism $A \to G \times G$ with the character $2 \text{tr} \pi \varphi$ is conjugate to $\varphi \times \varphi$. Taking $F = \overline{F}$ and $\psi := \psi' := \varphi \times \varphi$, we find $g = 1$ in Th. 6.2.2 as stated, but also $\tau \psi = \psi$ where $\tau \in \text{Aut}(G \times G)(\overline{F})$ interchanges the two summands.

To characterize the isomorphism criterion of Theorem 6.2.3 we need the following proposition.

### 6.4. Proposition

For each Killing-Cartan type $\Phi$ in the table

<table>
<thead>
<tr>
<th>Type $\Phi$</th>
<th>$F_4$</th>
<th>$G_2$</th>
<th>$E_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>3</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

there is a unique algebraic group $G_0$ of type $\Phi$ that is compact at every real place of every number field; it is defined over $\mathbb{Q}$. For every field $K$ of characteristic zero and $n$ as in the table, the following are equivalent:

1. $G_0 \otimes K$ is split.
and \( \zeta \)

Then there is an embedding \( A \) (resp. \( G \)) whose character is defined over \( F \) (resp. \( K \)). By [A, Theorem 9(3,4,5)] there is a representation \( \phi: A \rightarrow G(\mathbb{Q}) \) whose character \( \chi \) is defined over \( F = \mathbb{Q} \) (resp. \( F = \mathbb{Q}(\zeta_9 + \bar{\zeta}_9), F = \mathbb{Q}(\sqrt{13}) \)). Moreover, \( G \) acts transitively on the homomorphisms \( A \rightarrow G(\mathbb{Q}) \) with character \( \chi \) (see [A] and [Griess, Cor. 1 and 2]). By [A, 9.3(1)] the representation \( \varphi \) is irreducible, so \( Z_{G(\mathbb{Q})}(A) = 1 \). Thus, all conditions of Theorem 6.2 are satisfied. Therefore there is a twisted form \( G_0 \) of \( G \) defined over \( F \) and an embedding \( A \rightarrow G_0(F) \).

In particular, there is an embedding \( A \rightarrow G_0(\mathbb{R}) \). Since any finite subgroup of a Lie group is contained in its maximal compact subgroup, it is easy to see that \( G_0 \otimes_F \mathbb{R} \) is compact for all embeddings of \( F \) into \( \mathbb{R} \). Moreover, by Theorem 6.2 we have an embedding \( A \rightarrow G(K) \) iff \( G_0 \) and \( G \) are isomorphic over \( K \). By Proposition 6.4 the latter occurs iff \(-1\) is a sum of 4 squares of \( K \).

(Thus, we have recapitulated the argument from [Serre 00, 2.5.3]).

6.6. Example (type \( E_8 \)). Let \( G \) denote the split group of type \( E_8 \), \( A = G(\mathbb{F}_2) \) (resp. \( A = \text{PSL}(2, 8), \text{PSL}(2, 13) \)), \( K \) a field of characteristic zero. We view \( A \) as a subgroup of \( \text{GL}_{248} \) via the adjoint representation. There is an embedding \( A \rightarrow G(K) \) iff \(-1\) is a sum of 16 squares (resp. and \( \zeta_{11} + \bar{\zeta}_{11} \in K \) for \( A = \text{SL}(2, 32) \)). Indeed, by [GR, Theorem 2.27 and Theorem 3.25] there exists an embedding \( A \rightarrow G(\mathbb{Q}) \) whose character is defined over \( F = \mathbb{Q} \) (resp. \( F = \mathbb{Q}(\zeta_{11} + \bar{\zeta}_{11}) \)). Using [GR] one can check all conditions of Theorem 6.2 (cf. Example 6.5).

It follows by Theorem 6.2 that there is an embedding \( A \rightarrow G_0(F) \) for some twisted form \( G_0 \) of \( G \). Again as in Example 6.5 one can see that \( G_0 \) is the unique group such that \( G_0 \otimes_F \mathbb{R} \) is compact for all embeddings of \( F \) into \( \mathbb{R} \). Finally by Proposition 6.4 \( G \) and \( G_0 \) are isomorphic over a field
extension $K/F$ iff $-1$ is a sum of 16 squares in $K$. This proves Theorem 1.1.

Roughly speaking, we have added the facts about the compact $E_8$ contained in the proof of Proposition 6.4 (which uses the existence of the $u$-invariant) to Serre’s appendix [GR, App. B].

One can also take $G$ to be the form of $E_8$ over $\mathbb{Q}$ that is neither split nor anisotropic. Then in the same way one can show that $A$ embeds in $G(K)$ iff $-1$ is a sum of 4 squares (resp., and $\zeta_{11} + \bar{\zeta}_{11} \in K$ for $A = \text{SL}(2,32)$).

In the same way one can get the following example:

6.7. Example (type $A_2$). Let $G = \text{PGL}_2$, $A = \text{Alt}_4$ (resp. $\text{Alt}_5$), and $K$ a field of characteristic zero. Then there is an embedding $A \rightarrow G(K)$ iff $-1$ is a sum of 2 squares and for $\text{Alt}_5$ additionally $\sqrt{5} \in K$ (see [Serre 72, §2.5] and [Serre 80, §1]).

Acknowledgments. Both authors thank J-P. Serre for his helpful comments on an earlier version of this paper. The first author’s research was partially supported by the National Science Foundation grant no. DMS-0653502. The second author was supported by DFG, project GI706/1-1, and by the MPIM Bonn.

References


