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The luckiest strategy on earth? A better way to buy lottery scratch-off tickets

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Skip Garibaldi (skip@garibaldibros.com, MR ID 622970, 0000-0001-8924-5933) got interested in the lottery many years ago thanks to insightful questions asked by students in his finite probability class at UCLA. Since then, he left the professor track to work for IDA, a nonprofit corporation, where he currently serves as the director of IDA's Center for Communications Research in La Jolla, California. He occasionally buys lottery tickets.

A little over 10 years ago, an article in *Harper's*, [**Ric11**], declared Joan Ginther to be "the luckiest woman on earth" for having claimed four large lottery prizes. It was a good story. Dr. Ginther has a PhD from Stanford [**Gin76**] and was described as a "math professor" who "specialized in statistics". And she was well known in the small town in Texas where she grew up for coming back to visit, buying huge numbers of lottery scratch-off tickets, and being generous with the local residents. The central mystery of the story was: How did she win so much? People do get lucky, but is it plausible that someone could get lucky so many times?

Dr. Ginther's actions make it look like she wasn't just lucky. It looks like she had a plan and was working to execute it. A series of articles in the *Philadelphia Inquirer* examined a variety of strategies she may have pursued and deemed one to be most likely. Our aim here is to describe and analyze that strategy, which we can do using calculus.

Scratch-off tickets

You've probably seen lottery scratch-off tickets before. A typical US state lottery will have many different kinds of tickets, called *games*, for sale at one time. When a new game is created, the lottery operator prints a predetermined number of tickets, typically millions of them. All the tickets in a game cost the same amount, usually \$1 to \$20, although Texas introduced a game with \$100 tickets in May 2022. The price is printed on each ticket. Once a ticket is purchased, the buyer can scratch off an opaque waxy substance to reveal whether the ticket wins a prize and, if so, the amount of that prize. (Figure 1 shows some scratch-off tickets after scratching.) Whether a given ticket is a winner is determined at the start of the game, when the tickets are printed, but this information is only revealed after the ticket is purchased and scratched. The lottery operator expects to make money on the game, because the total amount of prizes to be given is quite a bit less than the total price to purchase all the tickets.

There is also another kind of game where the results of this paper apply, namely pull-tab games. These are smaller-scale versions of state-run scratch-off games [**Wik22, Min20**]. They are especially popular in Minnesota. These games are run by a bar operator or other enterprising businessperson, who buys a pre-printed box of

Figure 1. Some scratch-off tickets, after scratching

(typically) thousands of tickets and sells the tickets to their patrons. The operator pays out any prizes won by their customers. The operator expects to make money, because the cost to buy the box of tickets plus the value of the prizes is less than what the customers pay to buy the tickets from the operator. Tickets in these games are known as pull-tabs for historical reasons. Below we will mostly talk about scratch-off games and tickets, rather than pull-tabs, but the same analysis applies in both cases.

Basic strategy: buying the rest

Think now about just one scratch-off game. If you buy a bunch of tickets, typically you should expect to lose money, because the game is designed so that the lottery operator ("the house") will make money in the long run. It could happen, however, that other people buy a bunch of tickets and discover they are not winners, meaning that the rest of the tickets have more winners than expected among them. It is possible that the total value of the remaining prizes is greater than the cost to buy the rest of the tickets. In that situation, the average value of a ticket is greater than the cost of the ticket, and you could expect to make money by buying tickets. That can occur in real-life state lotteries, see for example [**CL20**].

If you had a way to know that such a thing had happened, you could use that as an indication for when to buy tickets yourself. In fact, many state lotteries, including the Texas Lottery, do provide information like that, meaning not just a list of all the prizes and how many total exist in the game, but also how many have been claimed. For state-run scratch-off games, you can find this information aggregated on third-party websites such as `scratchoff-odds.com`. In pull-tab games, the top prizes and how many have been claimed are listed on a wall poster that is displayed where the pull-tabs are sold. We can use this information to infer when the remaining tickets for sale are unusually likely to be winners. The website `scratchoff-odds.com` does exactly that, reporting a "score" for each game, which is how much you expect to win on each ticket on average divided by the cost of the ticket.

The basic strategy, then, is to wait for a game where people have bought up enough of the non-winning tickets that it is worth your while to buy all (or as many as you can) of the remaining tickets. For pull-tab games, buying all the remaining tickets is something gamblers discuss and do. For scratch-off games, it can be hard to execute this strategy, because there are so very many tickets, possibly spread throughout the state, that it is almost certainly impractical to purchase them all.

Better strategy: buy until you get a large prize

A better strategy, the one that journalist Peter Mucha described and advocated for in his *Philadelphia Inquirer* articles such as [**Muc14a**], is as follows. Suppose that, among the remaining unsold tickets, there are a few tickets each worth the same large amount, which is much larger than the other prizes available. The strategy is to buy tickets until you run out of money or you win a large prize, whichever comes first. Since this is alleged to be the strategy used by Ginther, "the luckiest woman on earth", one might call this strategy "the luckiest strategy on earth" as we have done in the title. That's a fairly silly name, though, so we will call it the *better* strategy.

The strategy itself is simple, and some gamblers would instinctively follow it. Mucha's claim is that the better strategy has a different range of outcomes from the basic strategy, and its outcomes are better. This is an interesting distinction because the published analyses we have seen are for the basic strategy, not the better strategy. Our goal in this note is to analyze the better strategy, and in particular to evaluate Mucha's claim. Spoiler alert: He is right.

Let's start by setting some notation. If it were possible for you to buy all the remaining tickets, you would: (a) pay the price to purchase the tickets, (b) claim all the large prizes, and (c) claim all the other prizes, meaning the prizes that are not "large". As a practical matter, in real-world games, doing just (a) and (c) would result in a net loss, and from now on we assume that to be true. Call the amount lost in that scenario 1 unit. Think of the value of one of the large prizes, j, as being some number of units, typically less than 1. If there are n large prizes remaining, then doing (a), (b), and (c) would net you $nj - 1$ units. On average, for every dollar you spend on tickets, you would net $nj - 1$ dollars.

That went by pretty quickly, and may not seem like much happened. But we have really done something. With this choice of notation, we have eliminated several variables from consideration, such as the cost of a ticket, the number of remaining tickets, and the values of all the prizes except for the large prizes.

Figure 2 illustrates this notation with data for two example scratch-off games. All but the bottom three lines consist of information that can be read off the state lottery website. We list nj, because for the basic strategy it is useful to compare nj to 1. We also give the value of $1/j$, because for the better strategy (as we shall see) it is useful to compare it to the integers $n - 1$, n, and $n + 1$.

A disclaimer is in order. The numbers in Figure 2 will suffice for our purpose, which is to provide concrete examples for our mathematics with an extra frisson of being based in reality. However, if you are thinking of using the numbers from `scratchoff-odds.com` or a state lottery (such as the ones reproduced in the figure) to guide your investments, please note the following deficiencies. First, the number provided (the number of unclaimed large prizes) is only an upper bound for the number we want to know (the number n of large-prize-winning tickets that have not yet been bought), because there may be people who have prize-winning tickets that have not yet claimed their prize. Second, you will likely have to pay taxes on any prizes that you win, meaning you will receive not j but a somewhat smaller amount for a large prize, thereby reducing your expected net. We will ignore these deficiencies in

Game	Fastest Road to $1,000,000	Big Money
State	Florida	Georgia
Date	August 2022	June 2023
Tickets remaining	652,565	43,610
Ticket price	$30	$25
Total of other prizes	$11,932,940	$694,890
Value of each large prize	$1,000,000	$2,500,000
Number n of large prizes	7	1
One unit	$7,644,010	$395,360
$1/j$ (compare to n)	7.64401	0.27796
nj (compare to 1)	0.91575	3.59769

Figure 2. Two examples of scratch-off games.

this paper, like the website `scratchoff-odds.com` and similar ones do.

Here is another approximation we will treat as an equality: We will say that buying a fraction f of the remaining tickets and claiming all the non-large prizes you win from those tickets would cost you f units. This isn't precisely correct, because the "other" prizes in (c) may not be evenly distributed between the tickets you purchase and those you don't. But in real games there are lots of other prizes and their value is small relative to the large prize, so this approximation is likely close to the true cost and we ignore this distinction.

Example 1 (one large prize). Returning to the abstract setting, suppose there is only one large prize remaining. In this case, the only difference between the better strategy and the basic strategy is that the "better" strategy does not buy the (losing) tickets that remain after you buy the large prize. This is obviously better than the basic strategy.

In formulas, the basic strategy nets $j - 1$. With the better strategy, you expected to find the large prize after buying half the remaining tickets, so you win j and spend $\frac{1}{2}$, netting $j - \frac{1}{2}$.

We can use the same reasoning to say something about the general case. Imagine the tickets as being in a gigantic stack that we will purchase starting from the top, and put X for the fraction of the way down the stack where the first large prize lies. It is 0 if the first big prize is at the top of the stack, and 1 if the prize is at the bottom. We don't know X, but that doesn't stop us from using it in our formulas. With this notation, in the better strategy you spend X and win j, netting $j - X$. Therefore:

Theorem 2. *The better strategy is expected to net $j - 1/(n+1)$. In particular, it has a positive expected net if and only if $1/j < n + 1$.*

To see this, it suffices to note that the average or expected value of X is $1/(n+1)$. This likely aligns with your intuition. (The fancy explanation is as follows: The location of each of the n large prizes is a standard uniform random variable and X is the minimum of these n variables, so it is a Beta$(1, n)$ random variable. Such a random variable has expected value of $1/(n+1)$.)

Looking at the two strategies from a different angle, with the basic strategy you win nj at a cost of 1 unit, for a "rate of return" of nj. With the better strategy, you win j with an average cost of $1/(n+1)$, for an average rate of return of $(n+1)j$.

Regardless of the values of n and j, $(n+1)j$ is greater than nj, which gives another argument that the so-called better strategy is indeed better than the basic strategy.

Example 3 (should I do it twice?). Suppose $1/j < n+1$, so you decide to pursue the better strategy and successfully claim a large prize. You might wonder: Should I do the better strategy a second time?

To address this question, suppose that your large prize is the only one that was claimed, so that the number of outstanding large prizes is $n-1$, and that during your first execution of the strategy a fraction F of the remaining tickets were purchased in total, by you and by other players. Then a "unit" is now $1-F$ and the large prize amount — which is unchanged in real-world currency — is $j/(1-F)$ in the new units. Consequently, Theorem 2 says that the better strategy has a positive expected net if and only if

$$nj > 1 - F. \tag{1}$$

From the original setting, we assumed that $(n+1)j > 1$. Substituting this into condition (1), we find a sufficient condition: *If $F \geq j$, then following the better strategy a second time has a positive expected net.* It is interesting that this criterion involves comparing F (a fraction of tickets sold) versus j (the value of a large prize) — the fact that comparing these two quantities is a sensible thing to do is a nice outcome of our notation.

Example 4 (very small f). Suppose you will only buy a very small number of tickets, perhaps a fraction f of the remaining tickets where f is very close to zero. That's true for most ticket buyers, people who aren't pursuing an investment strategy. In that setting, the two strategies have essentially the same expected net. That's because you either fail to win a large prize, in which case you net $-f$ in both strategies, or you succeed in winning one, in which case under either strategy you net somewhere between j and $j-f$, two numbers that are almost equal.

A more realistic model

Our analysis so far has omitted an important detail: in practice we will only be able to buy some fraction f of the tickets, whether because of how much money we have available or due to other practical limitations. Including this detail does not really change our analysis of the basic strategy. You would spend f units and expect to collect nf of the large prizes, netting $f(nj-1)$ with a rate of return of $nj-1$.

However, including this detail does change our analysis of the better strategy. Let's figure out $E(f)$, the expected net from pursuing the better strategy while buying at most f of the tickets. For example, at $f=0$, which represents the case where you do not buy any tickets, we have $E(0)=0$. The other extreme $f=1$ was the case we already analyzed where you plan to buy as many tickets as you must to collect a large prize, therefore (Theorem 2):

$$E(1) = j - 1/(n+1). \tag{2}$$

To understand what happens when $0 < f < 1$, let's use the random variable X from earlier. If $X \leq f$, then you will win a large prize and net $j - X$. If $X > f$, then you will not win a large prize, and you will net $-f$. Therefore, the expected net from the better strategy is

$$E(f) = \int_0^f (j - x)\beta(x)\,\mathrm{d}x + \int_f^1 -f\beta(x)\,\mathrm{d}x, \tag{3}$$

where $\beta(x)$ is the probability density function for the random variable X.

It is a fact from probability that $\beta(x) = n(1-x)^{n-1}$ for $0 \leq x \leq 1$. With this in hand, we could evaluate this integral, but the result is a complicated expression and is not very useful. Instead, let's understand $E(f)$ by plotting it using calculus.

We already know the values at the endpoints, the cases $f = 0$ or 1, so let us examine the derivative $E'(f)$ with respect to f, which we compute from the formula (3). For the first integral, we can apply the Fundamental Theorem of Calculus to see that its derivative is $(j - f)\beta(f)$. For the second integral, we pull the f outside the integral and apply the product rule for derivatives to see that the derivative is

$$-\int_f^1 \beta(x)\,\mathrm{d}x + f\beta(f) = -(1-f)^n + f\beta(f).$$

Putting this together, we find:

$$E'(f) = j\beta(f) - (1-f)^n = (1-f)^{n-1}(f - 1 + nj). \tag{4}$$

(Let's pause at this point and look back at the case of very small f, like for a casual ticket buyer. The rate of return for the better strategy is $E(f)/f$, and the rate of return for small f is $\lim_{f\to 0} E(f)/f$. This equals $\lim_{f\to 0} E'(f)$ by L'Hôpital's Rule. Using Equation (4), we see that it is $nj - 1$, which is the same rate of return as from the basic strategy, confirming the conclusion of Example 4.)

Returning to our project of plotting $E(f)$, we ask: For what values of f is $E'(f) = 0$? Trivially, the term $(1-f)^{n-1}$ in the product (4) is zero when $f = 1$ and $n > 1$. Otherwise, E' vanishes at $f^* := 1 - nj$, which belongs to the interval $(0, 1)$ if and only if $n < 1/j$.

Next let us consider the second derivative,

$$E''(f) = n(1-f)^{n-2}(1 - (n-1)j - f).$$

For what values of f is $E''(f) = 0$? Trivially, the term $(1-f)^{n-2}$ in the product is zero when $f = 1$ and $n > 2$. Otherwise, E'' vanishes at

$$f^{**} := 1 - (n-1)j = f^* + j,$$

which belongs to the interval $(0, 1)$ if and only if $n - 1 < 1/j$.

Three different cases

Because the vanishing of E' and E'' depends on the value of $1/j$, let's break up our analysis into cases based on that. The ticket buyer prefers j to be large and therefore $1/j$ to be small.

Best case. Suppose first that $1/j < n - 1$, the best case for the potential ticket buyer. In this case $E'(f)$ is positive and $E''(f)$ is negative for $f \in (0, 1)$, so the graph of $E(f)$ is increasing and concave down as exhibited in Figure 3a.

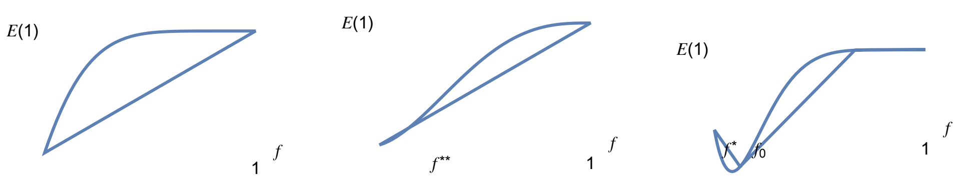
For a game like this, we have

$$nj > 1 + 1/(n - 1), \tag{5}$$

as can be seen by applying the inequality $1/j < n - 1$ twice. One might compare this to the sufficient condition $nj > 1$ for the basic strategy to have a positive expected net.

As part of preparing this article, I tracked quite a few lottery scratch-off games and did not see any that fell into this case. Here is a heuristic argument why that may be expected. In practice, games tend to have $nj < 1.15$. (The game "Big Money" from Figure 2 is a notable exception.) Combining this with (5), we deduce $n \geq 8$. At the same time, the fact that $nj - 1$ is positive means that some large amount of losing tickets have been sold, yet many (at least 8) large prizes still remain. This feels unlikely, because a game that starts with many large prizes has two reasons to not land in the $nj > 1$ regime, let alone (5): First, the large prizes must have relatively small values because there are a lot of them and the lottery operator expects to make money on the game. Second, in order to be in the state $nj > 1$, the large prizes must have been disproportionately located among the unsold tickets. The Law of Large Numbers suggests that this is less likely to occur if there are lots of large prizes.

Good case. Suppose next that $n - 1 < 1/j < n$, as is the case for the game "Big Money" from Figure 2. In this case, $E'(f)$ is positive on $(0, 1)$ whereas $E''(f)$ is positive to the left of f^{**} and negative to the right, as depicted in Figure 3b. In that plot, it looks like f^{**} is less than 1/2. It turns out that this is a consequence of how we generated the cartoon, which was for an example with $n = 3$. From the hypotheses on this case, $1 - nj$ is negative, so $f^{**} < j < 1/n$.



(a) Best case
$1/j < n - 1$

(b) Good case
$n - 1 < 1/j < n$

(c) Edge case
$n < 1/j < n + 1$

Figure 3. Plot of $E(f)$ for different values of n and j. Vertical axes are on different scales.

Edge case. Finally, suppose that $n < 1/j < n + 1$, as is the case for the game "Fastest Road to $1,000,000" from Figure 2. Of the three cases we consider, this is the least favorable to the potential ticket buyer and the most complicated. In this case, the better strategy has a positive expected net (Theorem 2) and the basic strategy has a negative expected net.

The derivative E' is negative left of f^* and positive to the right. It follows that the unique critical point f^* is a minimum of E and that the maximum of E occurs at one of the endpoints. Our hypothesis on the values of n and j guarantee that $E(1)$ is positive. Since $E(0) = 0$, the maximum of $E(f)$ on $[0, 1]$ is $E(1)$.

For concavity, we note that E'' is positive left of f^{**} and negative to the right. Thus the graph of $E(f)$ can be divided into three regions: On $(0, f^*)$, the function is decreasing (therefore negative) and concave up. On (f^*, f^{**}), it is increasing and concave up. On $(f^{**}, 1)$ it is increasing and concave down.

Since $E(f^*)$ is negative and $E(1)$ is positive, the graph of E must cross the horizontal axis at a point $f_0 \in (f^*, 1)$. And that is the only place it crosses the horizontal axis, by Rolle's Theorem since E' is not zero on $(f^*, 1)$. This is enough information to plot $E(f)$, see Figure 3c. The fact that f^* is so far to the left in the interval $[0, 1]$ is not a coincidence, because $f^* = 1 - (n + 1)j + j < j < 1/n$.

The value of f_0 is a key parameter: it is the minimum fraction of tickets you need to commit to buying in order to expect a non-negative net on this strategy. It can be tricky to give a closed-form expression for f_0, because n and j are indeterminates, so we won't pursue that further here. The good news is that, in any specific example, we can compute f_0 easily using a computer algebra system such as Mathematica.

Example 5. In the game "Fastest Road to \$1,000,000" from Figure 2, $E(1) = 0.00582$, meaning that if you are prepared to buy tickets until you hit a large prize, you can expect to net about \$45,000. In order to get the desired large prize, you expect to have to buy 1/8th, i.e., 12.5%, of the remaining tickets at a price of about \$2.45 million. Applying a computer algebra system to solve $E(f_0) = 0$, we find that you need to commit to buying not 12.5%, but at least $f_0 \approx 22\%$ of the tickets in order to have a positive expected value, to account for the possibility that the first large prize may appear later than average.

Because the graph of $E(f)$ flattens out for large f, you can capture most of the expected \$45,000 net by committing to buying just 60% of the tickets. However, that strategy comes with a large standard deviation of about 0.11 units or \$850,000.

This example may feel disappointing. Yet the story is actually a positive one. With previous analyses of ticket buying, which followed the basic strategy, you would conclude that the expected net is negative in the edge case. However, the expected net is positive with the better strategy. Since the edge case is perhaps the most common of the three cases to occur, changing strategies means that many more games now have a positive expected net.

Did she do it?

We have focused here on analyzing the better strategy, rather than the tantalizing mystery of Dr. Ginther's apparently successful method. If you are hungry for more on that subject, I recommend reading Mucha's series of articles.

One thing we can say for certain is that Ginther is well known for buying a lot of scratcher tickets, including focusing on tickets in specific games [**Muc14b**]. That is a pattern one would expect to see for someone who seriously pursues the basic or better strategy. And Ginther isn't the only person who has won large prizes after buying a remarkable number of scratch-off tickets. Newspaper stories such as [**EC20**] reported an "Ivy League group" called Black Swan Capital LLC doing just that in several states around 2020. Maybe they too were following the better strategy. Their observed actions are not inconsistent with such a conclusion.

Conclusion

In this paper, we have seen various concrete ways that the better strategy is better than the basic strategy, as claimed by Mucha. For example, the better strategy has a positive expected net in every game where the basic strategy does, and others besides (namely, games that fall in the edge case). Many ticket buyers might naturally follow the better strategy rather than the basic one, so the contribution here is the analysis of the better strategy using calculus and the discussion of the various outcomes. If you choose to exercise this strategy yourself, I wish you the best of luck!

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Summary. Lottery winner Dr. Joan Ginther, sometimes called “the luckiest woman on earth”, appeared to be pursuing a deliberate strategy in her lottery wins. Journalist Peter Mucha made a conjecture as to what her strategy might have been, and we analyze his proposal here. Using calculus, we show that the strategy proposed by Mucha is indeed better than the naive strategy. We give formulas showing how much a gambler pursuing this strategy can expect to make.

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