ON THE SPLIT EXCEPTIONAL LIE ALGEBRA E_7 *

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Introduction

Let J be the split exceptional Jordan algebra over the field F of characteristic $\neq 2,3$. That is, we shall regard J as the set of 3 by 3 matrices

$$a = \begin{pmatrix} \alpha_1 & a_{12} & a_{13} \\ \bar{a}_{12} & \alpha_2 & a_{23} \\ \bar{a}_{13} & \bar{a}_{23} & \alpha_3 \end{pmatrix},$$

where $\alpha_i \in F$, and where a_{ij}, $1 \leq i < j \leq 3$, are elements of the split Cayley algebra C based on F. Composition in J is defined by $a \cdot b = ab + ba$, the product ab denoting ordinary multiplication of matrices with entries in C. Addition and multiplication by scalars are defined in obvious fashion. For the sake of concreteness, we take C to be the algebra of 2 by 2 matrices

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \quad \alpha, \beta \in F, \quad a,b \in F^3, \text{ with involution } c \rightarrow \bar{c}$$

sending $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$ into $\begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix}$, and with product

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + (a,d) & \alpha c + \delta a - (b \wedge d) \\ \gamma b + \beta d + (a \wedge c) & \beta\delta + (b,c) \end{pmatrix},$$

where $a \wedge c$ and (a,c) denote respectively the customary vector and scalar products in F^3.

Then J carries a symmetric bilinear form $(a,b) = Tr(a \cdot b)$,

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and a symmetric trilinear form $(a,b,c) = \mathrm{Tr}((a\circ b)\circ c) - \mathrm{Tr}(a)(b,c) - \mathrm{Tr}(b)(a,c) - \mathrm{Tr}(c)(a,b) + 2\mathrm{Tr}(a)\mathrm{Tr}(b)\mathrm{Tr}(c)$ (see [8]). The form (a,b) is non-singular on J. Thus if a and b are elements of J, there is a unique element $a\times b$ in J satisfying $(a\times b,c) = (a,b,c)$ for all $c \in J$. In fact, one has

$$a\times b = a\circ b - \mathrm{Tr}(a)b - \mathrm{Tr}(b)a + (\mathrm{Tr}(a)\mathrm{Tr}(b) - \tfrac{1}{2}(a,b))I ,$$

where I is the identity matrix diag $\{1,1,1\}$ in J (thus $2^{-1}I$ is the identity of J). Clearly $a\times b = b\times a$.

Let V be the space of all 2 by 2 matrices of the form

$$A = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix},$$

$\alpha,\beta \in F$, $a,b \in J$, and define a quartic form q on V by

(1) $$q(A) = (a\times a, b\times b) - \tfrac{2}{3}(a,a,a)\alpha - \tfrac{2}{3}(b,b,b)\beta - 2(\tfrac{1}{2}(a,b) - \alpha\beta)^2 .$$

Polarization yields the following 4-linear form, for which $(A,A,A,A) = 24\,q(A)$; let

$$A_i = \begin{pmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{pmatrix}, \quad 1 \le i \le 4 ; \text{ then}$$

l.c. greek "pi" - π

(2) $$\begin{aligned}(A_1,A_2,A_3,A_4) = {} & 4\Sigma''_\pi(a_{1\pi}\times a_{2\pi}, b_{3\pi}\times b_{4\pi}) - 4\alpha_1(a_2,a_3,a_4) \\ & - 4\alpha_2(a_1,a_3,a_4) - 4\alpha_3(a_1,a_2,a_4) - 4\alpha_4(a_1,a_2,a_3) \\ & - 4\beta_1(b_2,b_3,b_4) - 4\beta_2(b_1,b_3,b_4) - 4\beta_3(b_1,b_2,b_4) \\ & - 4\beta_4(b_1,b_2,b_3) - \Sigma'_\pi(a_{1\pi},b_{3\pi})(a_{2\pi},b_{4\pi}) \\ & + 2\Sigma_\pi \alpha_{1\pi}\beta_{2\pi}(a_{3\pi},b_{4\pi}) - 8\Sigma''_\pi \alpha_{1\pi}\alpha_{2\pi}\beta_{3\pi}\beta_{4\pi},\end{aligned}$$

where Σ_π indicates that the sum is to be taken over all permutations π of $(1,2,3,4)$, Σ'_π that only those twelve permutations π with $1\pi < 2\pi$ are to be taken, and Σ''_π that only those six permutations π with $1\pi < 2\pi$ and $3\pi < 4\pi$ are to be taken.

Now let L be the set of all linear transformations T of V

such that $(A,A,A,AT) = 0$ for all $A \in V$, or, equivalently, such that $(A_1T,A_2,A_3,A_4) + (A_1,A_2T,A_3,A_4) + (A_1,A_2,A_3T,A_4) + (A_1,A_2,A_3,A_4T) = 0$ for all $A_i \in V$. Then L is a Lie algebra of linear transformations of V, and is closed under p-th powers if F is of prime characteristic p; hence L is a restricted Lie algebra in the latter case [7].

The Lie algebra L is adapted, for the case of a general field F, from a complex Lie algebra providing a realization of the 133-dimensional simple Lie algebra E_7 as an irreducible Lie algebra of linear transformations in a space of minimal dimension, namely 56. This realization of E_7 has been described by Freudenthal [2]. We propose to show here that the Lie algebra L defined above is a realization of the unique simple Lie algebra of classical type E_7 over F, in the sense of [5], and to apply general techniques [6-9] to the study of its automorphism group.

1. The Freudenthal resolution.

It will be recalled [8] that there is a realization of the simple Lie algebra of classical type E_6 over F as the set of linear transformations T of J such that $(aT,a,a) = 0$ for all $a \in J$; we call this algebra $E_6(J)$. There is also an "outer" automorphism of $E_6(J)$ given by $T \to -T^*$, where T^* is the adjoint of T with respect to the bilinear form (a,b) on J. Now $E_6(J)$ may be regarded as a subalgebra of L via the mapping sending $T \in E_6(J)$ onto the mapping which sends

$$A = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} 0 & aT \\ -bT^* & 0 \end{pmatrix} = A' .$$

This mapping of $E_6(J)$ into L is clearly one-one, linear, and preserves the Lie product. To see that $A \to A'$ is in L, we have

$$(A',A,A,A) = 12(aT\times a, b\times b) - 12(a\times a, bT^*\times b) - 12\alpha(aT,a,a)$$
$$+ 12\beta(bT^*,b,b) - 6(aT,b)(a,b) + 6(a,bT^*)(a,b)$$
$$+ 12\alpha\beta(aT,b) - 12\alpha\beta(a,bT^*) .$$

Since $(aT,a,a) = 0 = (bT^*,b,b)$ and $(aT,b) = (a,bT^*)$, the above reduces to $12[(aT\times a, b\times b) - (a\times a, bT^*\times b)]$; that is, we have $(A',A,A,A) = 12[(aT,a,b\times b) - (a\times a, bT^*,b)]$. By the skewness of T, T^* with respect to the (symmetric) trilinear form, this is $-6[(a,a,(b\times b)T) - ((a\times a)T^*,b,b)] = 6[((a\times a)T^*, b\times b) - (a\times a,(b\times b)T^*)] = 0$. Thus $A \to A'$ is in L, and these mappings, formed for all $T \in E_6(J)$, form a 78-dimensional subalgebra of L isomorphic to $E_6(J)$.

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in V$, $T \in L$, and $IT = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in V$.

Then from $0 = (I,I,I,IT) = -24(\alpha+\beta)$ we have $\beta = -\alpha$. Define a linear transformation S of V by

l.c. greek "xi" - ξ "eta" - η

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} S = \begin{pmatrix} \alpha\xi & -\frac{1}{3}\alpha x \\ \frac{1}{3}\alpha y & -\alpha\eta \end{pmatrix} , \quad \alpha \in F .$$

Then $I(T-S)$ has the form

$$\begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} .$$

One sees at once from (2) that $S \in L$, using only the bilinearity of $a\times b$ and the multilinearity of the forms involved. Upon replacing T by $T-S \in L$, we may assume $IT = I(T-S)$ as above.

Now let $P(c)$ be the mapping of V sending

$$A = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} \frac{1}{2}(b,c) & \beta c \\ c\times a & 0 \end{pmatrix} ,$$

$Q(d)$ the mapping sending A into

$$\begin{pmatrix} 0 & d\times b \\ \alpha d & \frac{1}{2}(a,d) \end{pmatrix} ,$$

where c and d are as above. Then each of these mappings is in $\underline{L}$, and subtracting theur sum from T gives a reduction to the case IT = 0. To establish that P(c) and Q(d) are in $\underline{L}$ involves a rather long calculation, which reduces to the following identity:

$$(3) \qquad 6(x \times x, y \times (z \times x)) = (y,z)(x,x,x) + 3(y,x)(z,x,x).$$

For once (3) is shown, let $A \in \underline{V}$ be as above; then

$$\begin{aligned}(A,A,A,AP(c)) &= 12(a \times a, b \times (c \times a)) + 12\beta(b \times b, a \times c) \\ &\quad - 12\alpha\beta(a,a,c) - 2(b,c)(a,a,a) - 12\beta(b,b,c \times a) \\ &\quad - 6(a,b)(a,c \times a) - 6\beta(a,b)(c,b) + 12\alpha\beta(a,c \times a) \\ &\quad + 12\alpha\beta^2(b,c) + 6\beta(b,c)(a,b) - 12\alpha\beta^2(b,c) \\ &= 12(a \times a, b \times (c \times a)) - 2(b,c)(a,a,a) - 6(a,b)(a,c \times a) \\ &= 2(6(a \times a, b \times (c \times a)) - (b,c)(a,a,a) - 3(a,b)(a,c,a)),\end{aligned}$$

which is zero by (3). A similar calculation shows that (3) implies that each $Q(d) \in \underline{L}$. We defer the proof of (3) to the next section.

Now suppose that T is in $\underline{L}$, and that IT = 0. Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \underline{V}, \quad HT = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}.$$

Then from 3(IT,I,I,H) + (I,I,I,HT) = 0 = (I,I,I,HT), we have $-24(\alpha + \beta) = 0$, or $\alpha + \beta = 0$. From (H,H,H,HT) = 0, we find $24(\alpha - \beta) = 0$, or $\alpha = \beta$. Thus $\alpha = \beta = 0$.

l.c. greek "phi" - φ "psi" - ψ

Now if $C = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \in \underline{V}$, then $CT = \begin{pmatrix} \varphi(c) & cR \\ cS & \psi(c) \end{pmatrix}$,

where φ and ψ are linear functions on $\underline{J}$, and R,S are linear transformations of $\underline{J}$, all of these being determined by T. Then $0 = 3(IT,I,I,C) + (I,I,I,CT)$ yields $\psi = -\varphi$; further, $0 = (IT,C,C,C) + 3(I,C,C,CT)$ gives $0 = (I,C,C,CT) = -4(c,c,cR)$, so that $R \in \underline{E}_6(\underline{J})$. Finally, $0 = (C,C,C,CT) = -4\varphi(c)(c,c,c)$. Since both $\varphi(c)$ and (c,c,c) are defined by polynomials in the coefficients of c relative to some basis for J, these polynomials being homogeneous of respective degrees 1 and 3, and since (c,c,c) is not identically zero, it follows that $\Phi(c) = 0$ for all c (F has at least 5 elements). Thus $\Phi = 0 = \Psi$; similarly, if

$$D = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} \in V, \text{ then } DT = \begin{pmatrix} 0 & dU \\ dW & 0 \end{pmatrix},$$

where $W \in E_6(J)$ depends only on T, as does the transformation U of J.

With C and HT as above, we have $0 = 3(HT,H,H,C) + (H,H,H,CT)$, from which $0 = (HT,H,H,C) = -4(b,c)$ for all $c \in J$. Thus $b = 0$, and $a = 0$ (in HT) is a similar consequence of $0 = 3(HT,H,H,D) + (H,H,H,DT)$; hence we have $HT = 0$.

From $0 = 2(IT,I,C,D) + (I,I,CT,D) + (I,I,C,DT) = (I,I,CT,D) + (I,I,C,DT)$, one has $0 = 4(cR,d) + 4(c,dW)$. Thus $W = -R^*$, the image of R under the exceptional automorphism of $E_6(J)$.

Finally, from $0 = (IT,C,C,D) + 2(I,CT,C,D) + (I,C,C,DT)$ we see that $-4(c,c,dU) = 0$ for all $c,d \in J$, from which it follows by polarization on c that $(c,e,dU) = 0$ for all $c,d,e \in J$. In particular, $0 = (I,I,dU) = 4Tr(dU)$, so that $Tr(dU) = 0$. Then for all $e \in J$, we have $0 = (I,e,dU) = -(e,dU)$, so that $dU = 0$ by the non-singularity of the trace form. Likewise $S = 0$, and we now have established that

$$\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} T = \begin{pmatrix} 0 & cR \\ -dR^* & 0 \end{pmatrix}, \quad R \in E_6(J) .$$

By analogy with the method of Freudenthal [2] in the complex case, we have therefore shown that every $T \in L$ is a sum of transformations of four types considered above, all of which are in L, and which are collected below:

$$(4)\qquad \begin{aligned}
N_\alpha &: \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} \alpha\xi & -\frac{1}{3}\alpha x \\ \frac{1}{3}\alpha y & -\alpha\eta \end{pmatrix}, \quad \alpha \in F ; \\
P_a &: \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}(a,y) & \eta a \\ a \times x & 0 \end{pmatrix}, \quad a \in J ; \\
Q_a &: \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} 0 & a \times y \\ \xi a & \frac{1}{2}(a,x) \end{pmatrix}, \quad a \in J ; \\
T_R &: \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} 0 & xR \\ -yR^* & 0 \end{pmatrix}, \quad R \in E_6(J).
\end{aligned}$$

We thus have a linear mapping $(\alpha,a,b,R) \to N_\alpha + P_a + Q_b + T_R$ from the direct sum $F \oplus J \oplus J \oplus E_6(J)$ onto L. This mapping is in fact one-one; for if $N_\alpha + P_a + Q_b + T_R = 0$, the image of I is

$$\begin{pmatrix} \alpha & a \\ b & -\alpha \end{pmatrix} = 0 ,$$

so that $\alpha = 0$, $a = 0 = b$, and hence $T_R = 0$, from which $R = 0$. Thus L has dimension $1 + 27 + 27 + 78 = 133$, and may be regarded as the space $F \oplus J \oplus J \oplus E_6(J)$, with the composition $[(\alpha,a,b,R),(\gamma,c,d,S)] = (\delta,e,f,U)$, where

$$(5)\qquad \begin{aligned}
\delta &= \tfrac{1}{2}\,((b,c) - (a,d)), \\
e &= \tfrac{2}{3}\,(\gamma a - \alpha c) + aS - cR , \\
f &= \tfrac{2}{3}\,(\alpha d - \gamma b) + dR^* - bS^*, \\
xU &= \tfrac{1}{6}((b,c) - (a,d))x + \tfrac{1}{2}((b,x)c - (x,d)a) + d \times (a \times x) \\
&\quad - b \times (c \times x) + x[RS] , \quad x \in J.
\end{aligned}$$

(This product may be read off from the effect of the commutator $[N_\alpha+P_a+Q_b+T_R\ ,\ N_\gamma+P_c+Q_d+T_S]$ on I and on $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$.)

2. Proof of the identity (3).

We shall use in the proof of this identity several other identities in J, which it will be helpful to list (such evident identities as $x \cdot y = y \cdot x$ are omitted):

(6) $((x \cdot x) \cdot y) \cdot x = (x \cdot x) \cdot (y \cdot x)$ (Jordan identity);

(7) $(x, y \cdot z) = (x \cdot y, z)$;

(8) If $Tr(w) = 0$, then

$(x \cdot w, y, z) + (x, y \cdot w, z) + (x, y, z \cdot w) = 0$;

(9) $(x \cdot x) \cdot x = 2Tr(x) x \cdot x - (2Tr(x)^2 - (x,x))x + \frac{1}{3}(x,x,x)I$.

l.c. greek "lambda" - λ

Now every element of J can be written in the form $\lambda I + b$, where $Tr(b) = 0$. In view of the bilinearity of (3) in y and z, it thus suffices to prove (3) in the four cases: (i) $y = z = I$; (ii) $y = I$, $Tr(z) = 0$; (iii) $z = I$, $Tr(y) = 0$; (iv) $Tr(y) = 0 = Tr(z)$.

In case (i), we have $I \times (I \times x) = I \times (Tr(x)I - x) = x + Tr(x)I$, from which $6(x \times x, I \times (I \times x)) = 6(x \times x, x) + 6Tr(x)(x \times x, I)$ $= 6(x,x,x) + 3(I,x)(x,x,I) = (I,I)(x,x,x) + 3(I,x)(I,x,x)$, and (3) is established.

In case (ii), we have $I \times (z \times x) = Tr(z \times x)I - (z \times x)$ $= (z,x)I - \frac{3}{2}(z,x)I - z \cdot x + Tr(x)z + \frac{1}{2}(z,x)I = - z \cdot x + Tr(x)z$, from which $6(x \times x, I \times (z \times x)) = -6(x \times x, x \cdot z) + 6Tr(x)(x \times x, z)$ $= -6(x,x,x \cdot z) + 6Tr(x)(x,x,z)$. The first term here is zero by (8), and the second is $3(I,x)(z,x,x)$. Since $(I,z) = 2Tr(z) = 0$, we have $(I,z)(x,x,x) = 0$, and (3) again holds.

In case (iii), $y \times (I \times x) = y \times (Tr(x)I - x) = -Tr(x)y - x \cdot y$ $+ Tr(x)y + \frac{1}{2}(x,y)I = - x \cdot y + \frac{1}{2}(x,y)I$. Thus $6(x \times x, y \times (I \times x))$ $= -6(x,x,x \cdot y) + 3(x,y)(x,x,I) = (y,I)(x,x,x) + 3(x,y)(I,x,x)$ as in (ii).

In case (iv) we note first that $(x\times x, y\times(z\times x))$ $= (x\times x, y, z\times x) = ((x\times x)\times(z\times x), y)$, and that $(x\times x)\times(z\times x)$

$$= (x\circ x - 2\mathrm{Tr}(x)x + (\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))I)\times(z\circ x - \mathrm{Tr}(x)z - \tfrac{1}{2}(z,x)I)$$

$$= (x\circ x)\times(z\circ x) - 2\mathrm{Tr}(x)(x\times(z\circ x)) + (\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))(I\times(z\circ x))$$

$$- \mathrm{Tr}(x)(x\circ x)\times z + 2\mathrm{Tr}(x)^2(x\times z) + \mathrm{Tr}(x)(\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))z$$

$$-\tfrac{1}{2}(z,x)(x\circ x)\times I + \mathrm{Tr}(x)(z,x)x\times I - \tfrac{1}{2}(z,x)(\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))I\times I$$

$$= (x\circ x)\times(z\circ x) - 2\mathrm{Tr}(x)(x\times(z\circ x)) + (z,x)(\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))I$$

$$-(\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))z\circ x - \mathrm{Tr}(x)(x\circ x)\times z + 2\mathrm{Tr}(x)^2(x\times z)$$

$$+(\mathrm{Tr}(x)^3 - \tfrac{1}{2}\mathrm{Tr}(x)(x,x))z - \tfrac{1}{2}(x,x)(z,x)I + \tfrac{1}{2}(z,x)x\circ x$$

$$+ (z,x)\mathrm{Tr}(x)^2 I - \mathrm{Tr}(x)(z,x)x - (z,x)(\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))I$$

$$= (x\circ x)\times(x\circ z) - 2\mathrm{Tr}(x)(x\times(x\circ z)) + (z,x)\mathrm{Tr}(x)^2 I - \tfrac{1}{2}(z,x)(x,x)I$$

$$- \mathrm{Tr}(x)(x\circ x)\times z - (\mathrm{Tr}(x)^2 - \tfrac{1}{2}(x,x))z\circ x + 2\mathrm{Tr}(x)^2 x\times z$$

$$+ (\mathrm{Tr}(x)^3 - \tfrac{1}{2}\mathrm{Tr}(x)(x,x))z + \tfrac{1}{2}(z,x)x\circ x - \mathrm{Tr}(x)(z,x)x \,.$$

We use the above to compute $6(x\times x, y\times(z\times x)) =$ $6((x\times x)\times(z\times x), y)$; in the expansion of this scalar product using our last value for $(x\times x)\times(z\times x)$, the first two terms are $6((x\circ x)\times(x\circ z), y) - 12\mathrm{Tr}(x)(x\times(x\circ z), y) = 6(x\circ x, x\circ z, y) - 12\mathrm{Tr}(x)(x, x\circ z, y)$, which since $\mathrm{Tr}(z) = 0$ is equal by (8) to $-6((x\circ x)\circ z, x, y)$ $- 6(x\circ x, x, y\circ z) + 6\mathrm{Tr}(x)(x, x, y\circ z)$. Since $\mathrm{Tr}(y) = 0$, we have $(I, y) = 0$; making use of these substitutions, we find

$$6(x\times x, y\times(z\times x)) = -6((x\circ x)\circ z, x, y) - 6(x\circ x, x, y\circ z)$$

$$+ 6\mathrm{Tr}(x)(x, x, y\circ z) - 6\mathrm{Tr}(x)(x\circ x, z, y)$$

(10) $$- 6\mathrm{Tr}(x)^2(z\circ x, y) + 3(x,x)(z\circ x, y)$$

$$+ 12\mathrm{Tr}(x)^2(x, z, y) + 6\mathrm{Tr}(x)^3(z, y)$$

$$- 3\mathrm{Tr}(x)(x,x)(z,y) + 3(z,x)(x\circ x,y)$$

$$- 6\mathrm{Tr}(x)(z,x)(x,y) .$$

Now $((x\circ x)\circ z,x,y) = (((x\circ x)\circ z)\times x,y) = (((x\circ x)\circ z)\circ x,y)$ $- ((x\circ x),z)(x,y) - \mathrm{Tr}(x)((x\circ x)\circ z,y) = ((x\circ x)\circ(z\circ x),y) - (x\circ x,z)(x,y)$ $- \mathrm{Tr}(x)((x\circ x)\circ z,y)$ by (6), and this is

$$((x\circ x)\times(z\circ x),y) + (x,x)(z\circ x,y) + (z,x)(x\circ x,y)$$
$$- (x,y)(x\circ x,z) - \mathrm{Tr}(x)((x\circ x)\circ z,y)$$
$$= (x\circ x,x\circ z,y) + (x,x)(z\circ x,y) + (z,x)(x\circ x,y)$$
$$- (x,y)(x\circ x,z) - \mathrm{Tr}(x)((x\circ x)\circ z,y)$$
$$= -((x\circ x)\circ z,x,y) - (x\circ x,x,y\circ z) + (x,x)(z\circ x,y)$$
$$+ (z,x)(x\circ x,y) - (x,y)(x\circ x,z) - \mathrm{Tr}(x)((x\circ x)\circ z,y) ,$$

again using (8). Since this is an expression for $((x\circ x)\circ z,x,y)$, we have

$$((x\circ x)\circ z,x,y) = \tfrac{1}{2}[-(x\circ x,x,y\circ z) + (x,x)(z\circ x,y) + (z,x)(x\circ x,y) - (x,y)(x\circ x,z) - \mathrm{Tr}(x)((x\circ x)\circ z,y)]. \tag{11}$$

Furthermore, $(x\circ x,x,y\circ z) = ((x\circ x)\times x,y,z)$, and $(x\circ x)\times x$ $= (x\circ x)\circ x - (x,x)x - \mathrm{Tr}(x)x\circ x + (\mathrm{Tr}(x)(x,x) - \tfrac{1}{2}(x\circ x,x))I$ $= \mathrm{Tr}(x)x\circ x - 2\mathrm{Tr}(x)^2x + (\tfrac{1}{3}(x,x,x) + \mathrm{Tr}(x)(x,x) - \tfrac{1}{2}(x\circ x,x))I$ by (9); thus

$$(x\circ x,x,y\circ z) = \mathrm{Tr}(x)(x\circ x,y\circ z) - 2\mathrm{Tr}(x)^2(x,y\circ z) + \tfrac{2}{3}(x,x,x)(y,z) + 2\mathrm{Tr}(x)(x,x)(y,z) - (x\circ x,x)(y,z) . \tag{12}$$

Substitution in (10) of (11) and (12) yields

$$6(x\times x,y\times(z\times x)) = -3\mathrm{Tr}(x)(x\circ x,y\circ z) + 6\mathrm{Tr}(x)^2(x,y\circ z) - 2(y,z)(x,x,x) - 9\mathrm{Tr}(x)(x,x)(y,z) + 3(y,z)(x\circ x,x) + 3(x,y)(x\circ x,z) + 3\mathrm{Tr}(x)((x\circ x)\circ z,y) + 6\mathrm{Tr}(x)(x,x,y\circ z) \tag{13}$$

$$- 6\mathrm{Tr}(x)(x\circ x,z,y) - 6\mathrm{Tr}(x)^2(z\circ x,y) + 12\mathrm{Tr}(x)^2(x,z,y)$$
$$+ 6\mathrm{Tr}(x)^3(y,z) - 6\mathrm{Tr}(x)(x,z)(x,y) .$$

By (7) and the commutativity of J, the first and seventh terms cancel, as do the second and tenth. It will also be noted that the eighth and ninth terms combine to give

$$6\mathrm{Tr}(x)[(x \times x,y\circ z) - (x\circ x,y \times z)]$$
$$= 6\mathrm{Tr}(x)[(x\circ x,y\circ z) - 2\mathrm{Tr}(x)(x,y\circ z) + 2\mathrm{Tr}(x)^2(y,z)$$
$$- (x,x)(y,z) - (x\circ x,y\circ z) + (x,x)(y,z)]$$
$$= - 12\mathrm{Tr}(x)^2(x,y\circ z) + 12\mathrm{Tr}(x)^3(y,z) .$$

The fifth term of (13) is equal to $3(y,z)(x \times x,x) + 6\mathrm{Tr}(x)(x,x)(y,z) - 6\mathrm{Tr}(x)^3(y,z) + 3\mathrm{Tr}(x)(x,x)(y,z)$, of which expression the first term is $3(y,z)(x,x,x)$. Now substitution in (13) from the above yields

$$6(x \times x,y \times (z \times x)) = (y,z)(x,x,x) + 12\mathrm{Tr}(x)^3(y,z) + 3(x,y)(x\circ x,z)$$
(14)
$$- 12\mathrm{Tr}(x)^2(x,y\circ z) + 12\mathrm{Tr}(x)^2(x,z,y)$$
$$- 6\mathrm{Tr}(x)(x,y)(x,z) .$$

Since $(x\circ x,z) = (x,x,z) + 2\mathrm{Tr}(x)(x,z)$, while $(x,y\circ z) = (x,y,z) + \frac{1}{2}(y,z)(x,I) = (x,y,z) + \mathrm{Tr}(x)(y,z)$, we may substitute in (14) to get $6(x \times x,y \times (z \times x)) = (y,z)(x,x,x) + 3(x,y)(x,x,z)$, which completes case (iv) and the proof of the identity (3).

3. Identification of L as E_7 .

From (5) one sees that $[(1,0,0,0),(0,0,0,S)] = 0$ for all $S \in E_6(J)$, so that if H_0 is a standard Cartan subalgebra [6] of $E_6(J)$, then $H = (0,0,0,H_0) + (F,0,0,0)$ is a seven-dimensional commutative subalgebra of $\underline{L}$. Also if S is a root-vector relative to H_0 in $\underline{E}_6(J)$, then $(0,0,0,S)$ belongs to a weight $\underline{\alpha}$ of the adjoint representation of $\underline{H}$ in $\underline{L}$, such that $\underline{\alpha}((1,0,0,0)) = 0$. Moreover, one sees from (5) that

$$[(0,a,0,0),(0,0,0,T)] = (0,aT,0,0) ,$$

$$[(0,a,0,0),(1,0,0,0)] = \tfrac{2}{3}(0,a,0,0) ,$$

$$[(0,0,b,0),(0,0,0,T)] = -(0,0,bT^*,0) ,$$

$$[(0,0,b,0),(1,0,0,0)] = -\tfrac{2}{3}(0,0,b,0) ,$$

$$[(0,0,b,0),(0,a,0,0)] = (\underline{\alpha},0,0,U) ,$$

where $\underline{\alpha} = \frac{1}{2}(a,b)$, $cU = \frac{1}{6}(a,b)c + \frac{1}{2}(b,c)a - b \times (a \times c)$.

Letting $u_1, \ldots ,u_8$ be the basis for C we have used in studying $E_6(J)$ [8], and letting $u_i(j,k) = u_i E_{jk} + \bar{u}_i E_{kj} \in J$, $1 \le i \le 8$, $1 \le j < k \le 3$, these $u_i(j,k)$ together with the matrix units E_{jj}, $1 \le j \le 3$, form a basis for J relative to which the transformations in a certain Cartan subalgebra of $E_6(J)$ act diagonally [8]. To show that $\underline{L}$ is simple of type E_7, we may assume that H_0 is this Cartan subalgebra. From [8], H_0 has a basis $h_1, \ldots ,h_6$ with

$$u_i(1,2)h_j = \delta_{ij}u_i(1,2) , \; 1 \le i,j \le 4 ;$$

$$u_{i+4}(1,2)h_j = -\delta_{ij}u_{i+4}(1,2) , \; 1 \le i,j \le 4 ;$$

$$u_k(1,2)h_5 = 0 , \; u_k(1,2)h_6 = u_k(1,2) , \; 1 \le k \le 8 .$$

Since $[(0,u_k(1,2),0,0),(1,0,0,0)] = \frac{2}{3}(0,u_k(1,2),0,0)$, each $(0,u_k(1,2),0,0)$, $1 \le k \le 8$, is a root-vector in $\underline{L}$ relative to $\underline{H}$. The corresponding root $\underline{\alpha}_k(1,2)$ is given by

$$\alpha_k(1,2)(\Sigma_{j=1}^{7} \lambda_j h_j) = \lambda_k + \lambda_6 + \tfrac{2}{3}\lambda_7 \ , \ 1 \le k \le 4 \ ;$$

$$\alpha_k(1,2)(\Sigma_{j=1}^{7} \lambda_j h_j) = -\lambda_{k-4} + \lambda_6 + \tfrac{2}{3}\lambda_7 \ , \ 5 \le k \le 8 \ ;$$

here $h_i = (0,0,0,h_i)$ $1 \le i \le 6$, $h_7 = (1,0,0,0)$. Similarly $(0,0,u_k(1,2),0)$ belongs to the root $\beta_k(1,2)$, where

$$\beta_k(1,2)(\Sigma \lambda_j h_j) = \lambda_k - \lambda_6 - \tfrac{2}{3}\lambda_7 \ , \ 1 \le k \le 4 \ ;$$

$$\beta_k(1,2)(\Sigma \lambda_j h_j) = -\lambda_{k-4} - \lambda_6 - \tfrac{2}{3}\lambda_7 \ , \ 5 \le k \le 8 \ .$$

Likewise, results on $E_6(J)$ from [8] show that :

$(0,u_k(1,3),0,0)$ belongs to $\alpha_k(1,3)$, where

$$\alpha_k(1,3)(\Sigma \lambda_j h_j) = \tfrac{1}{2}(2\lambda_k - \Sigma_{j=1}^{4} \lambda_j) + \lambda_5 - \lambda_6 + \tfrac{2}{3}\lambda_7 \ , \ 1 \le k \le 4 \ ;$$

$$\alpha_k(1,3)(\Sigma \lambda_j h_j) = \tfrac{1}{2}(\Sigma_{j=1}^{4} \lambda_j - 2\lambda_{k-4}) + \lambda_5 - \lambda_6 + \tfrac{2}{3}\lambda_7 \ , \ 5 \le k \le 8 \ ;$$

$(0,0,u_k(1,3),0)$ belongs to $\beta_k(1,3)$, where

$$\beta_k(1,3)(\Sigma \lambda_j h_j) = \alpha_k(1,3)(\Sigma_{j=1}^{4} \lambda_j h_j) - \lambda_5 + \lambda_6 - \tfrac{2}{3}\lambda_7, \ 1 \le k \le 8 \ ;$$

$(0,u_k(2,3),0,0)$ belongs to $\alpha_k(2,3)$, where

$$\alpha_k(2,3)(\Sigma \lambda_j h_j) = \tfrac{1}{2}(\lambda_4 + 2\lambda_k - \Sigma_{j=1}^{3} \lambda_j) - \lambda_5 + \tfrac{2}{3}\lambda_7 \ , \ 1 \le k \le 3 \ ;$$

$$\alpha_4(2,3)(\Sigma \lambda_j h_j) = -\tfrac{1}{2}\Sigma_{j=1}^{4} \lambda_j - \lambda_5 + \tfrac{2}{3}\lambda_7 \ ;$$

$$\alpha_k(2,3)(\Sigma \lambda_j h_j) = \tfrac{1}{2}(\Sigma_{j=1}^{3} \lambda_j - \lambda_4 - 2\lambda_{k-4}) - \lambda_5 + \tfrac{2}{3}\lambda_7 \ , \ 5 \le k \le 7 \ ;$$

$$\alpha_8(2,3)(\Sigma \lambda_j h_j) = \tfrac{1}{2}\Sigma_{j=1}^{4} \lambda_j - \lambda_5 + \tfrac{2}{3}\lambda_7 \ ;$$

$(0,0,u_k(2,3),0)$ belongs to $\beta_k(2,3)$, where

$$\beta_k(2,3)(\Sigma \lambda_j h_j) = \alpha_k(2,3)(\Sigma_{j=1}^{4} \lambda_j h_j) + \lambda_5 - \tfrac{2}{3}\lambda_7 \ , \ 1 \le k \le 8 \ ;$$

$(0,E_{ii},0,0)$ belongs to γ_i, $1 \le i \le 3$, where

$$\gamma_1(\Sigma \lambda_j h_j) = 2\lambda_5 + \tfrac{2}{3}\lambda_7 \ , \quad \gamma_2(\Sigma \lambda_j h_j) = -2\lambda_5 + 2\lambda_6 + \tfrac{2}{3}\lambda_7 \ ,$$

$$\gamma_3(\Sigma \lambda_j h_j) = -2\lambda_6 + \tfrac{2}{3}\lambda_7 \ ;$$

$(0,0,E_{ii},0)$ belongs to $-\gamma_i$, $1 \leq i \leq 3$.

All the $\alpha_k(j,m)$, $\beta_k(j,m)$, γ_i, and all the non-zero roots to which belong vectors of the form $(0,0,0,S)$ are distinct, span the dual space of H, and contain no arithmetic progression of roots whose difference is a root and having length greater than three (in fact the only progressions of this form of length three have the form $-\alpha$, 0, α , where 0 is admitted as a root). One now sees that L has center zero, and has all the characteristic properties of a Lie algebra of classical type [5] except possibly the properties $[LL] = L$ and that $[L_\alpha L_{-\alpha}]$ be one-dimensional for every non-zero root α (for H and the root-vectors listed above span L). To show $[LL] = L$, we see by the above and the fact that $E_6(J)$ is its own derived algebra that it suffices to show $(1,0,0,0) \in [LL]$. But now from $[(0,0,b,0),(0,a,0,0)] = \frac{1}{2}((a,b),0,0,U)$, $U \in E_6(J)$ it follows that $(1,0,0,0) \in [LL]$, and from $\beta_i(j,k) = -\alpha_{i+4}(j,k)$, $\beta_{i+4}(j,k) = -\alpha_i(j,k)$, $1 \leq i \leq 4$, $1 \leq j < k \leq 3$, together with $(u_i(j,k),u_{i+4}(j,k)) = 4$, $1 \leq i \leq 4$, $1 \leq j < k \leq 3$, and $(E_{ii},E_{ii}) = 2$, $1 \leq i \leq 3$, that $[L_\alpha L_{-\alpha}] \neq 0$ for all roots α whose root-vectors are not in $E_6(J)$. For those $L_\alpha \subseteq E_6(J)$, we already know that $[L_\alpha L_{-\alpha}] \neq 0$ from [8]. Since all L_α are one-dimensional, it follows that all $[L_\alpha L_{-\alpha}]$ are. Hence L is of classical type, therefore a direct sum of simple algebras of classical type [5]. We display a fundamental system of roots of type E_7 for L ; from this the simplicity of L follows by [5].

From [8], we take a fundamental system of roots in $E_6(J)$ with respect to H_0, calling them $\alpha_2, \ldots ,\alpha_7$, and extend these to H by $\alpha_i(h_7) = 0, \ 2 \leq i \leq 7$. We adjoin the root $\alpha_1 = \gamma_1$. With this choice, we then have:

$$\alpha_1(\Sigma\ \lambda_j h_j) = 2\lambda_5 + \frac{2}{3}\lambda_7 \ ;$$

$$\alpha_2(\Sigma\ \lambda_j h_j) = -\lambda_1 - 2\lambda_5 + \lambda_6 \ ;$$

$$\alpha_3(\Sigma\ \lambda_j h_j) = \lambda_1 - \lambda_2 \ ;$$

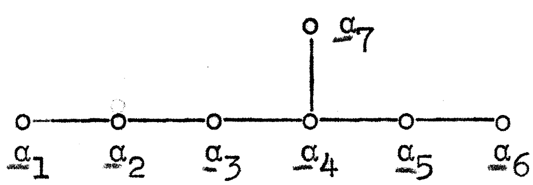
$$\alpha_4(\Sigma\ \lambda_j h_j) = \lambda_2 - \lambda_3 \ ;$$

$$\alpha_5(\Sigma\ \lambda_j h_j) = \lambda_3 + \lambda_4 \ ;$$

$$\alpha_6(\Sigma\ \lambda_j h_j) = \frac{1}{2}\Sigma_{j=1}^{4}\ \lambda_j + \lambda_5 - 2\lambda_6 \ ;$$

$$\alpha_7(\Sigma\ \lambda_j h_j) = \lambda_3 - \lambda_4 \ .$$

(The root-vectors corresponding to the last six of these will be found in [8].) They have the diagram [5]:



That is, they form a simple system of roots of type E_7. Therefore L is simple of this type, again by [5].

4. The enveloping algebra of L .

In this section we show that the enveloping associative algebra of L, as a Lie algebra of linear transformations of V, is the full algebra of linear transformations. By familiar results on associative algebras ([3], pp. 35,39), it suffices to show that L acts irreducibly in V and that its centralizer consists of scalar multiplications. To see the latter assertion, we first note that since $\pm 1, \pm \frac{1}{3}$ are all distinct in F, any linear transformation X of V which commutes with all N_α of (4) must have the form l.c. greek "mu" - μ

$$\begin{pmatrix}\xi & x\\ y & \eta\end{pmatrix} X = \begin{pmatrix}\lambda\xi & xS\\ yT & \mu\eta\end{pmatrix},$$

where λ, μ are fixed in F, and where S, T map J into J. If, moreover, X commutes with all P_a of (4), we have

$$\begin{pmatrix}\frac{1}{2}\lambda(a,y) & \eta aS\\ (a \times x)T & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}(a,yT) & \mu\eta a\\ a \times (xS) & 0\end{pmatrix}$$

for all η, a, x, y ; from this it follows that $yT = \lambda y$, $aS = \mu a$ for all y, all a, and that $\lambda(a \times x) = a \times (\mu x) = \mu(a \times x)$ for all a, all x, from which $\mu = \lambda$, and $X = \lambda I$, a scalar.

To show that L acts irreducibly in V, let

$$v = \begin{pmatrix}\xi & x\\ y & \eta\end{pmatrix}$$

be a non-zero element of V. If $\eta \neq 0$, then a suitable $P_a P_b$ maps v into an element of the form

$$\begin{pmatrix}\xi & 0\\ y & 0\end{pmatrix}, \quad y \neq 0 ;$$

since $E_6(J)$ acts irreducibly in J [8], operation with an element T_R yields a non-zero element of $\begin{pmatrix}0 & 0\\ J & 0\end{pmatrix}$. If v has $x \neq 0$, then operating first with Q_b, $(x,b) \neq 0$, yields an element with $\eta \neq 0$, and we transform this into a non-zero element of $\begin{pmatrix}0 & 0\\ J & 0\end{pmatrix}$ as above. If $y \neq 0$ in the original v, we choose $a \in J$ with $a \times y \neq 0$ and operate with Q_a to obtain an element satisfying the condition $x \neq 0$. Finally, if $\xi \neq 0$, we choose $a \neq 0$ and operate with Q_a to obtain an element with $y \neq 0$. Thus every non-zero L-invariant subspace of J contains non-zero elements of $\begin{pmatrix}0 & 0\\ J & 0\end{pmatrix}$.

By the irreducibility of $E_6(J)$, any invariant subspace of V relative to L contains the subspace $\begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}$. Similarly, $\begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}$ is contained in every invariant subspace. Applying mappings P_a, Q_b now shows that V is irreducible relative to L.

5. Automorphisms of L.

Let K be the algebraic closure of F. Then the Lie algebra L_K obtained from L by extension of the base field acts in the space V_K, and is the algebra of all K-linear transformations of V_K which are skew with respect to the unique 4-linear extension of our form on V. By the above, L_K is a simple Lie algebra of type E_7 over K. We regard L as an F-subalgebra of L_K, V as an L-submodule of V_K. By [6], the automorphism group of L_K consists of "invariant" automorphisms.

Steinberg [9] has shown that the group I of invariant automorphisms of L (or of L_K) is generated by the mappings $\exp(\mathrm{ad}(\lambda e_\alpha))$, where e_α runs through a set of root-vectors relative to a given standard Cartan subalgebra. In particular, one may take as Cartan subalgebra our H of §3 (or H_K, in the case of L_K). From [8] it is known that if $T \in E_6(J)$ is a root-vector relative to H_0, then so is T^*, and $T^2 = 0 = T^{*2}$, acting in J. It follows at once that $(0,0,0,T)^2 = 0$, acting in V. Next we note that if $c \in J$ is among the $u_i(j,k)$ or the E_{ii}, then $c \times c = 0$; for in the former case, $c \circ c = 0$ and $\mathrm{Tr}(c) = 0$, while in the latter, $c \circ c = 2c$, $\mathrm{Tr}(c) = 1$, $(c,c) = 2$. Hence the root-vectors relative to H of the forms $(0,c,0,0)$, $(0,0,c,0)$ as listed in §3, which send $A = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$ into, respectively,

$$\begin{pmatrix} \frac{1}{2}(b,c) & \beta c \\ c \times a & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & c \times b \\ \alpha c & \frac{1}{2}(a,c) \end{pmatrix}$$

have squares sending A into

$$\begin{pmatrix} \frac{1}{2}(c \times a,c) & 0 \\ \beta(c \times c) & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \alpha(c \times c) \\ 0 & \frac{1}{2}(c \times b,c) \end{pmatrix} ,$$

respectively, and thus have squares equal to zero (for $(c \times a,c) = (c \times c,a) = 0$). Thus I is generated by mappings $\exp(\mathrm{ad}(\lambda e_\alpha))$, $e_\alpha \in L$, $e_\alpha^2 = 0$. It follows by §1 of [7] that the mappings $\exp(\lambda e_\alpha) = I + \lambda e_\alpha$ of V preserve the form q and that, for $T \in L$, $T \exp(\mathrm{ad}(\lambda e_\alpha)) = \exp(\lambda e_\alpha)^{-1} T \exp(\lambda e_\alpha)$. Thus the group of invariant automorphisms of L is contained in the group of automorphisms of the form $X \to U^{-1}XU$, where $q(AU) = q(A)$ for all $A \in V$.

The above considerations apply equally well in the algebraically closed case, where the fact that the invariant automorphisms constitute the full automorphism group yields the conclusion that the automorphism group of L_K consists of all mappings $X \to U^{-1}XU$, where U preserves the quartic form q on V_K. As in [7] and [8] we can now draw the following conclusions about the automorphisms of L :

<u>Theorem</u>. Let L be as above. Then the group of automorphisms of L is the set of mappings $X \to B^{-1}XB$, where B is a non-singular linear transformation of V such that for some $\beta(B) \neq 0$ in F and for all $A \in V$, $q(AB) = \beta(B)q(A)$. Such transformations of V constitute a group, the group of q-<u>similitudes</u>, and the group of automorphisms of L is isomorphic to the quotient group of the q-similitudes modulo scalar transformations.

6. q-Isometries and the group of Chevalley.

Let T be the group of transformations of V generated by the $I + \lambda e_\alpha$, $\lambda \in F$, α a root with respect to H. Then the center of T consists of scalars, and is the kernel of the mapping of T into the automorphism group $A(L)$ of L which sends $T \in T$ onto the automorphism $X \longrightarrow T^{-1}XT$ of L. The image is the group G' of Chevalley [1],[9]. Let S be the group of q-similitudes of V; then we have seen that the mapping sending $S \in S$ onto $X \longrightarrow S^{-1}XS$ is a homomorphism of S onto $A(L)$ with kernel F^*. Let Q be the group of q-preserving linear transformations of V, PQ its image in $A(L)$ under the above mapping, PT the image of T. Then $T \subseteq Q \subseteq S$, $PT \subseteq PQ \subseteq A(L)$, $PT = G'$ as above.

l.c. greek "omega" - ω

For $1 \leq i \leq 7$, let h_i be the unique element of $[L_{\alpha_i} L_{-\alpha_i}]$ with $\alpha_i(h_i) = 2$. If $\omega_i(h_j) = \delta_{ij}$, $1 \leq i,j \leq 7$, defines a basis $\{\omega_i\}$ for the dual space H^* dual to the basis $\{h_i\}$ for H, we have $\alpha_k = \Sigma_i\, \alpha_k(h_i)\omega_i = \Sigma_i\, A_{ki}\omega_i$, where A_{ki} is the (rational) Cartan integer corresponding to the pair of roots α_k, α_i. Then $A_{ki} = 0, -1$, or 2, the last only if $k = i$. Let P be the free abelian group on generators ω'_i in 1-1 correspondence with the ω_i. Let R be the subgroup of P generated by the elements $\alpha'_k = \Sigma_i\, A_{ki}\omega'_i$, $1 \leq i \leq 7$. Since (A_{ki}) is non-singular, the α'_k are free generators for R.

l.c. greek "chi" - χ

l.c. greek "theta" - θ

Corresponding to any F^*-valued character χ of R, there is (e.g., by the "isomorphism theorem" of [5]) a unique automorphism $\theta = \theta(\chi)$ of L leaving H fixed and with $e_{\alpha_i}\theta = \chi(\alpha'_i)e_{\alpha_i}$, $1 \leq i \leq 7$. If $\alpha = \Sigma\, m_i\alpha_i$ is the "canonical representation" [6] of a root α of L relative to $\alpha_1, \ldots, \alpha_7$, then

$$e_{\underline{\alpha}}{}^{\theta} = \left(\prod_i \chi(\underline{\alpha}'_i)^{m_i}\right) e_{\underline{\alpha}}$$

for $e_{\underline{\alpha}} \in L_{\underline{\alpha}}$.

Moreover, Chevalley [1] and Steinberg [9] have shown that every automorphism of $\underline{L}$ (of type E_7) is congruent, modulo the invariant subgroup G', to such an automorphism $\theta(\chi)$, and that $\theta(\chi)$ is in G' if and only if χ is the restriction to R of an $\underline{F}^*$-valued character of $\underline{P}$.

The choice of indices which we have adopted and the Cartan matrix for E_7 give:

$$\underline{\alpha}'_1 = 2\underline{\omega}'_1 - \underline{\omega}'_2 \; ,$$

$$\underline{\alpha}'_6 = 2\underline{\omega}'_6 - \underline{\omega}'_5 \; ,$$

$$\underline{\alpha}'_7 = 2\underline{\omega}'_7 - \underline{\omega}'_4 \; ,$$

$$\underline{\alpha}'_4 = -\underline{\omega}'_3 + 2\underline{\omega}'_4 - \underline{\omega}'_5 - \underline{\omega}'_7 \; ,$$

$$\underline{\alpha}'_i = -\underline{\omega}'_{i-1} + 2\underline{\omega}'_i - \underline{\omega}'_{i+1} \; , \; i = 2,3,5 \; .$$

The corresponding fundamental system of roots $\underline{\alpha}_1, \ldots, \underline{\alpha}_7$ is given by the same combinations of the $\underline{\omega}_i$. Now let

$$\underline{\beta}'_1 = -\underline{\omega}'_1 - 12\underline{\omega}'_4 + 24\underline{\omega}'_7 \; ;$$

$$\underline{\beta}'_2 = \underline{\omega}'_1 + \underline{\omega}'_2 - \underline{\omega}'_3 = \underline{\alpha}'_1 + \underline{\alpha}'_2 \; ;$$

$$\underline{\beta}'_3 = 5\underline{\omega}'_2 - 4\underline{\omega}'_3 - 24\underline{\omega}'_4 + 48\underline{\omega}'_7 = -5\underline{\alpha}'_3 - 6\underline{\alpha}'_4 - 4\underline{\alpha}'_5 - 2\underline{\alpha}'_6 + 21\underline{\alpha}'_7 \; ;$$

$$\underline{\beta}'_4 = \underline{\omega}'_2 - \underline{\omega}'_3 - 4\underline{\omega}'_4 - \underline{\omega}'_6 + 9\underline{\omega}'_7 = -\underline{\alpha}'_3 - \underline{\alpha}'_4 - \underline{\alpha}'_5 - \underline{\alpha}'_6 + 4\underline{\alpha}'_7 \; ;$$

$$\underline{\beta}'_5 = 3\underline{\omega}'_6 - 2\underline{\omega}'_7 = \underline{\alpha}'_5 + 2\underline{\alpha}'_6 - \underline{\alpha}'_7 \; ;$$

$$\underline{\beta}'_6 = 2\underline{\omega}'_6 - \underline{\omega}'_5 = \underline{\alpha}'_6 \; ;$$

$$\underline{\beta}'_7 = 2\underline{\omega}'_7 - \underline{\omega}'_4 = \underline{\alpha}'_7 \; .$$

Then $2\underline{\beta}'_1 = -3\underline{\alpha}'_1 - 4\underline{\alpha}'_2 - 5\underline{\alpha}'_3 - 6\underline{\alpha}'_4 - 4\underline{\alpha}'_5 - 2\underline{\alpha}'_6 + 21\underline{\alpha}'_7$, and one readily checks that $\beta'_1, \ldots, \beta'_7$ form a basis for P, while $2\beta'_1, \beta'_2, \ldots, \beta'_7$ form a basis for R. Thus $\theta = \theta(\chi)$ is in G' if and only if $\chi(2\beta'_1)$ is a square, i.e., if and only if

$$\chi(\alpha'_1)\chi(\alpha'_3)\chi(\alpha'_7) \in F^{*2}.$$

Now let χ_1 be a character of P defined as follows, relative to the character χ of R :

$$\chi_1(\beta'_i) = \chi(\beta'_i), \; 3 \le i \le 7 \;;$$

$$\chi_1(\beta'_2) = \lambda\chi(\alpha'_2); \; \chi_1(\beta'_1) = \rho \;;$$

here $\lambda \in F^*$ is so chosen that $\lambda\chi(\alpha'_3)^{-1}\chi(\alpha'_7) \in F^{*2}$, and $\rho \in F^*$ satisfies $\rho^2 = \lambda\chi(\alpha'_3)^{-1}\chi(\alpha'_7)(\lambda\chi(\alpha'_2))^{-4}\chi(\alpha'_3)^{-4}\chi(\alpha'_4)^{-6}\chi(\alpha'_5)^{-4}$ $\chi(\alpha'_6)^{-2}\chi(\alpha'_7)^{20}$. Then we have $\theta(\chi_1) \in G'$ by the above. Letting $\chi_2 = \chi\chi_1^{-1}$, a character of R , we see that $\theta(\chi_2) = \theta(\chi)\theta(\chi_1)^{-1} \in G'$ if and only if $\theta(\chi) \in G'$. Moreover, we have $\chi_2(\beta'_i) = 1$, $3 \le i \le 7$, $\chi_2(\beta'_2) = \chi(\beta'_2)\lambda^{-1}\chi(\alpha'_2)^{-1} = \chi(\alpha'_1)\lambda^{-1}$, and $\chi_2(2\beta'_1) = \chi(2\beta'_1)\rho^{-2} = \chi(\alpha'_1)^{-3}\lambda^3$, from the definition of χ_1. From $\chi_2(\beta'_7) = 1 = \chi_2(\beta'_6)$ we have $\chi_2(\alpha'_7) = 1 = \chi_2(\alpha'_6)$, and from these and $\chi_2(\beta'_5) = 1$, that $\chi_2(\alpha'_5) = 1$. Next, using these observations, we have $1 = \chi_2(\beta'_4) = \chi_2(\alpha'_3)^{-1}\chi_2(\alpha'_4)^{-1}$ and $1 = \chi_2(\beta'_3) = (\chi_2(\alpha'_3)\chi_2(\alpha'_4))^{-6}\chi_2(\alpha'_3)$, from which $\chi_2(\alpha'_3) = 1 = \chi_2(\alpha'_4)$. We then have

$$\chi_2(\beta'_2) = \chi_2(\alpha'_1)\chi_2(\alpha'_2) = \lambda^{-1}\chi(\alpha'_1) \;,$$

$$\chi_2(2\beta'_1) = \chi_2(\alpha'_1)^{-3}\chi_2(\alpha'_2)^{-4} = (\lambda^{-1}\chi(\alpha'_1))^{-3} \;,$$

from which $\chi_2(\alpha'_2) = 1$, $\chi_2(\alpha'_1) = \lambda^{-1}\chi(\alpha'_1)$. Thus $\chi_2(\alpha'_1)\chi_2(\alpha'_3)\chi_2(\alpha'_7) = \chi_2(\alpha'_1)$, and $\theta(\chi_2)$ (or $\theta(\chi)$) is in G' if and only if $\chi_2(\alpha'_1) \in F^{*2}$.

Now let S be a q-similitude of V, and consider the automorphism $X \to S^{-1}XS$ of L; by the remarks above, there exists $T \in T$ such that the automorphism $X \to (ST)^{-1}XST$ of L leaves fixed H and all root-vectors e_{α_i}, $e_{-\alpha_i}$, $2 \leq i \leq 7$, and multiplies e_{α_1} by $\lambda_1 \in F^*$. Moreover, the automorphism induced by S is induced by an element of T (*i.e.*, is in the group G') if and only if $\lambda_1 \in F^{*2}$. We further have $q(v(ST)) = q((vS)T) = q(vS) = \sigma q(v)$ for all $v \in V$, where $\sigma \in F^*$ is the factor associated with the q-similitude S. From our realization of L and the above it follows at once that the automorphism of L induced by $U = ST \in S$ leaves fixed the subalgebra $E_6(J)$ consisting of all mappings

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} 0 & xR \\ -yR^* & 0 \end{pmatrix}, \quad R \in E_6(J),$$

as well as the mapping N_α of (4) (since $N_\alpha \in H$).

As in § 4, it follows from $UN_\alpha = N_\alpha U$ that U has the form

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} \beta\xi & xB \\ yC & \gamma\eta \end{pmatrix},$$

where B and C are non-singular transformations of J and where β, γ are fixed nonzero scalars. The fact that U commutes with all elements of $E_6(J)$ and the absolute irreducibility of $E_6(J)$ in J now yield $B = \delta I$, $C = \varepsilon I$, where $\delta, \varepsilon \in F^*$. Thus U sends

l.c. greek "epsilon" - ε

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} \beta\xi & \delta x \\ \varepsilon y & \gamma\eta \end{pmatrix}.$$

Choosing $x = 0 = y$, with $\xi\eta \neq 0$, the condition $q(vU) = \sigma q(v)$ gives $(\beta\gamma)^2(\xi\eta)^2 = \sigma(\xi\eta)^2$, or $\sigma = (\beta\gamma)^2$. Thus $\sigma \in F^{*2}$, or <u>every factor of similitude is a square</u>.

Similarly, choosing $\eta = 0$, $y = 0$, $\xi \neq 0$, $(x,x,x) \neq 0$ gives $\beta\delta^3 = \sigma = (\beta\gamma)^2$ from the above; likewise, $\sigma = \gamma\varepsilon^3 = (\delta\varepsilon)^2$. Thus $\gamma = \sigma\varepsilon^{-3} = \mu^2\varepsilon^{-3}$, $\delta = \mu\varepsilon^{-1}$, where μ is one of the elements of F with $\mu^2 = \sigma$, and $\beta = \sigma\delta^{-3} = \mu^2\mu^{-3}\varepsilon^3 = \mu^{-1}\varepsilon^3$. Therefore

$$(15) \qquad \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} U = \begin{pmatrix} \mu^{-1}\varepsilon^3\xi & \mu\varepsilon^{-1}x \\ \varepsilon y & \mu^2\varepsilon^{-3}\eta \end{pmatrix},$$

$\varepsilon \neq 0$, $\mu^2 = \sigma$. Conversely, such a mapping U is a similitude with factor σ.

Now the root-space L_{α_1} in our decomposition is spanned by $(0,E_{11},0,0)$, where $E_{11} = \mathrm{diag}\{1,0,0\} \in J$, and where $(0,E_{11},0,0)$ sends

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} \frac{1}{2}(E_{11},y) & \eta E_{11} \\ E_{11} \times x & 0 \end{pmatrix}.$$

Thus $U^{-1}(0,E_{11},0,0)U = \lambda_1(0,E_{11},0,0)$, with the automorphism induced by U lying in G' if and only if $\lambda_1 \in F^{*2}$. Now $U^{-1}(0,E_{11},0,0)U$ sends

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} \frac{1}{2}\mu^{-1}\varepsilon^2(E_{11},y) & \mu^{-1}\varepsilon^2\eta E_{11} \\ \mu^{-1}\varepsilon^2(E_{11} \times x) & 0 \end{pmatrix},$$

so that $\lambda_1 = \mu^{-1}\varepsilon^2$, $\lambda_1 \in F^{*2}$ if and only if $\mu \in F^{*2}$.

We thus conclude that if S is a q-similitude with $q(vS) = \sigma q(v)$ for all $v \in V$, then $\sigma \in F^{*2}$, and there is a $T \in T$ with

$S = UT$, where U is as in (15); the automorphism of L induced by S (or that induced by U) is in G' if and only if $\mu \in F^{*2}$.

In particular, consider the case $S \in Q$, i.e., $\sigma = 1$. Then $S = UT$, where U is as in (15), with $\mu^2 = 1$. With $\mu = 1$, the automorphism induced by S is in G'; with $\mu = -1$, this automorphism is in G' if and only if $-1 \in F^{*2}$. Thus the mapping $I_S G' \to \mu F^{*2}$, sending the coset by G' of the automorphism I_S induced by S onto the coset μF^{*2}, yields an isomorphism of PQ/PT onto the quotient $(F^{*2} \cup (-1)F^{*2})/F^{*2}$. Applied to PS/PT, the corresponding mapping is an isomorphism of this group onto F^*/F^{*2}, an observation which may be viewed as another interpretation of the isomorphism $A(L)/G' = F^*/F^{*2}$ of Steinberg. It is clear that $PS/PQ = F^{*2}/F^{*4}$ under the mapping sending the coset corresponding to $S \in S$ into σF^{*4}, where $q(vS) = \sigma q(v)$ for all $v \in V$. We thus have the commutative diagram with exact rows:

$$(16) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & PQ/PT & \longrightarrow & PS/PT & \longrightarrow & PS/PQ & \longrightarrow & 1 \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \\ 1 & \longrightarrow & (F^{*2} \cup (-1)F^{*2})/F^{*2} & \longrightarrow & F^*/F^{*2} & \longrightarrow & F^{*2}/F^{*4} & \longrightarrow & 1 \end{array}$$

where the columns are the isomorphisms above, the mapping of PS/PT onto PS/PQ is the canonical homomorphism, and that of F^*/F^{*2} onto F^{*2}/F^{*4} sends λF^{*2} onto $\lambda^2 F^{*4}$.

As to the structure of S itself, rather than that of PS, we cannot be so precise. From the above it is clear that every $S \in S$ can be written in the form $S = \lambda UT$, where $\lambda \in F^*$, $T \in T$, and where U is as in (15) with $\mu^2 = \sigma$, the factor of similitude of S. Now T is an invariant subgroup of S, since whenever $S \in S$ and $e_\alpha \in L$ is a root-vector relative to a standard Cartan subalgebra, so is $S^{-1}e_\alpha S$. Because T can be described as the group generated by the $\exp(e_\alpha) = I + e_\alpha$ for all such $e_\alpha \in L$, we have the assertion. Furhtermore, S/T is commutative; for if $S_1 = \lambda_1 U_1 T_1$, $S_2 = \lambda_2 U_2 T_2$ are as above, we have $S_1 S_2 T = \lambda_1 \lambda_2 U_1 U_2 T$. It is clear from the form of U_1, U_2 that these commute, so that $S_1 S_2 T = S_2 S_1 T$. Hence T contains the commutator subgroup of S. We show that T is its own commutator subgroup, hence coincides with those of S and of Q. To this end, it suffices to show that each $I + e_\alpha$ is in the commutator subgroup of T, (by [9] we may assume that the e_α are root-vectors in our special decomposition, so that $e_\alpha^2 = 0$).

Let $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$ as above, and let M be the three-dimensional subalgebra of L spanned by e_α, $e_{-\alpha}$, $[e_\alpha e_{-\alpha}]$. By a result of Jacobson [4], the representation of M in V is completely reducible and V is the direct sum of trivial one-dimensional M-submodules and of irreducible two-dimensional submodules, in each of which it may be assumed that the matrices of e_α, $e_{-\alpha}$, $[e_\alpha e_{-\alpha}]$ are, respectively,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Now for all $\nu \neq 0$ in F, T contains the transformation $(I + \nu e_\alpha)(I + \nu^{-1} e_{-\alpha})(I + \nu e_\alpha)(I - e_\alpha)(I - e_{-\alpha})(I - e_\alpha)$. This mapping has in the 2-dimensional M-submodules of V the matrix diag $\{\nu, \nu^{-1}\}$, while it is of course the identity on the trivial submodules. Choosing ν such that $\nu^2 \neq 1$ and λ such that $\lambda(1 - \nu^{-2}) = 1$, we see that the commutator subgroup of T contains the transformation which is the identity on the trivial M-submodules of V and on the 2-dimensional submodules has matrix

$$\begin{pmatrix} \nu^{-1} & 0 \\ 0 & \nu \end{pmatrix}\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

that is, $I + e_\alpha$ is a commutator in T, as asserted.

A more concisely described set of generators for T may be given; namely, for each $c \in J$ the maps

$$(17)\quad \varphi_c: \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} \xi + \frac{1}{2}(y,c) + \frac{1}{4}(x,c,c) + \frac{1}{12}\eta(c,c,c) & x + \eta c \\ y + (x \times c) + \frac{1}{2}\eta(c \times c) & \eta \end{pmatrix}$$

and

$$(18)\quad \psi_c: \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \to \begin{pmatrix} \xi & x + (y \times c) + \frac{1}{2}\xi(c \times c) \\ y + \xi c & \eta + \frac{1}{2}(x,c) + \frac{1}{4}(y,c,c) + \frac{1}{12}\xi(c,c,c) \end{pmatrix}$$

are in T and the set of these mappings, formed for all $c \in J$, generates T. To see this, it will be noted from the multilinearity and symmetry of the forms involved that $c \to \varphi_c$ and $c \to \psi_c$ are homomorphisms of the additive group of J into the full linear group of V. When $c \in J$ satisfies $c \times c = 0$, we have $\varphi_c = I + (0,c,0,0)$, $\psi_c = I + (0,0,c,0)$, in the notation of § 1. By §§ 3,5, J has a basis consisting of elements c with $c \times c = 0$ and such that $(0,c,0,0)$, $(0,0,c,0)$ are root-vectors relative to a standard Cartan subalgebra of L. For each such c and each $\lambda \in F$, $\varphi_{\lambda c} = I + \lambda(0,c,0,0)$ and $\psi_{\lambda c} = I + \lambda(0,0,c,0)$ have the form $I + \lambda e_\alpha$, hence are in T. That φ_c, ψ_c are in T for all $c \in J$ now follows from the homomorphism property of the mappings $c \to \varphi_c$, $c \to \psi_c$.

To show that the φ_c and ψ_c generate T, it now suffices to show that the group they generate contains all $I + \lambda e_\alpha$, where e_α is one of the root-vectors of § 3. When e_α has the form $(0,c,0,0)$

or $(0,0,c,0)$, this has already been established. From § 3, our group contains all $I + \lambda e_\alpha$, where the coefficient of α_1 in α is different from zero. Thus the group of automorphisms of L induced by the group generated by the φ_c, ψ_c contains an automorphism of L which maps H into H and maps root-spaces according to the Weyl reflection S_α of the roots, for each α as above (cf. [6]). In particular, L_{α_2} and $L_{\alpha_1+\alpha_2}$ are conjugate by an element of our group which induces S_{α_1}, from which it follows that all $I + \lambda e_{\alpha_2}$ are in our group. A similar argument shows that the group in question contains all $I + \lambda e_\alpha$, $\alpha = \pm \alpha_i$, $1 \leq i \leq 7$. Since these S_{α_i} generate the entire Weyl group W, and since W acts transitively on the roots, another application of the argument yields the assertion.

In summary, we have the following theorem:

Theorem. Let L, V be as in § 1. Then the group S of q-similitudes of V is the set of non-singular linear transformations B of V such that for some $\beta(B)$ in F^{*2} and for all $v \in V$, $q(vB) = \beta(B)q(v)$. The commutator subgroup T of S is its own commutator subgroup, and T is properly contained in Q, the group of q-preserving transformations, if -1 is not a square in F. The group T is generated by the mappings φ_c, ψ_c of (17),(18), where c runs over J, and induces the simple Chevalley group G' of automorphisms of L. The groups of automorphisms of L induced by S, Q, T, respectively, are the quotients PS, PQ, PT of these by their centers, and these are related to the arithmetic of F as in the diagram (16).

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